A LOWER BOUND FOR A MEASURE OF PERFORMANCE OF AN EMPIRICAL BAYES ESTIMATOR

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We derive a lower bound for the measure of performance of an empirical Bayes estimator for the scale parameter θ of a Pareto distribution by using a weighted squared-error loss function and assuming a prior distribution uniform on (0,1).

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1. INTRODUCTION

Liang [2] and Tiwari and Zalkikar [5] proved some results concerning the performance of the empirical Bayes estimator of the scale parameter of Pareto distribution using a squared error loss function. In fact, Liang [2] relaxed the conditions imposed by Tiwari and Zalkikar [5] on the prior distribution and found a rate of convergence of order $n^{-\frac{2}{3}}$ – as compared to $n^{-\frac{1}{2}}$ found in [5] – for the sequence of empirical Bayes estimator.

In [4], a weighted squared-error loss function was considered and an empirical Bayes estimator for the scale parameter of Pareto distribution was proposed. There was assumed that the weights are given by a function which satisfies certain properties. It was proved that under certain condition, the empirical Bayes estimator of the scale parameter is asymptotic optimal and the corresponding rate of convergence is of order $n^{-\frac{2}{3}}$.

In this paper we derive a lower bound for the difference between the overall Bayes risk of the sequence of empirical Bayes estimators of the scale parameter of a Pareto distribution found in [4], and the Bayes risk of Bayes estimator of the same parameter. This difference is taken as a measure of performance of the empirical Bayes estimators. The loss function we use here is of weighted quadratic type.

The paper is structured as follows. In Section 2 there is described the Pareto distribution with a known shape parameter α and unknown scale parameter θ. Furthermore, the conditions that have to be satisfied in order to obtain the results from Sections 4 are stated. We define the Bayes risk for a weighted squared-error loss and the overall Bayes risk for a sequence of empirical Bayes estimators. Next, asymptotic optimality and rate of convergence for a sequence of empirical Bayes estimators are defined. Moreover, we recall some results of Preda and Ciumara [4] that we will apply in order to obtain the main result of this paper.

In Section 3, considering a uniform distibution for the prior, we find a lower bound for the difference between overall Bayes risk of the sequence of empirical Bayes estimators and the the Bayes risk of Bayes estimator.

The result obtained here generalizes the result of Liang [2] in the sense that if the weights function is constant and equal to unity, then we recover the case presented there. In fact, if the weights function satisfies certain properties, then the lower bound is the same as in the squared-error loss function case.

2. PRELIMINARIES

We consider a random variable X having a Pareto distribution for a given θ. The parameter θ is a value of a random variable Θ that has a prior distribution function $G : (0, \infty) \to [0, 1]$. Then the probability density function of $X|\Theta = \theta$ is

$$f\left(x|\theta\right) = \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}},$$

where $x > \theta$, $\alpha > 0$ and $\theta > 0$. The shape parameter α is known while the scale parameter θ is unknown. The marginal density of X is

$$f(x) = \int_0^{\min(x,m)} f\left(x|\theta\right) \mathrm{d}G(\theta) = \int_0^{\min(x,m)} f\left(x|\theta\right) g(\theta)\mathrm{d}\theta,$$

where $\mathrm{d}G(\theta) = g(\theta)\mathrm{d}\theta$.

If $\varphi(\theta) = \alpha\theta^{\alpha}$ and $u\left(x\right) = \frac{1}{x^{\alpha+1}}$, then $f\left(x|\theta\right) = \varphi(\theta)u(x)$ and

$$f(x) = u(x) \int_0^{\min(x,m)} \alpha\theta^{\alpha}\mathrm{d}G(\theta).$$

We impose (see[2]) the following conditions on the prior distribution G:

(A1) $G\left(m\right) = 1$ for some known positive real number m.

(A2) If $a^* = \sup\left\{\theta \mid G(\theta) = 0\right\}$, then f is a decreasing function in x on $(a^*, m]$.

As in [4], we consider the weighted squared-error loss $L : \mathbf{R}_+^2 \to \mathbf{R}_+$ defined as $L\left(x, \theta\right) = w(\theta)\left(x - \theta\right)^2$, with the weights function $w : \mathbf{R}_+ \to \mathbf{R}_+^*$ continuous and differentiable.

The conditions (see[4]) that w has to satisfy are

(A3) $\exists\, c_1 \in \mathbf{R}_+^*$ such that $w(\theta) \le c_1$, $\forall \theta \in \mathbf{R}_+$.
(A4) $\exists\, c_2 \in \mathbf{R}_+^*$ such that $0 \le w(\theta) + \theta w'(\theta) \le c_2$, $\forall \theta \in \mathbf{R}_+$ and $\exists\, \varepsilon > 0$ such that $w(\theta) + \theta w'(\theta) > \varepsilon$ on $(0, m]$.
(A5) $\exists\, \varepsilon_0 > 0$ such that $\varepsilon_0 < q(x) = E(w(\Theta)|X = x)$, $\forall x \in (0, m]$.

Then the Bayes estimator of θ given $X = x$ is

$$\varphi_G(x) = \frac{E(\Theta w(\Theta)|X = x)}{E(w(\Theta)|X = x)}, \tag{2.1}$$

assuming that all posterior expectations involved in the above expression exist and $E(w(\Theta)|X = x) \neq 0$.

The Bayes risk of φ_G is

$$R(G, \varphi_G) = E\left(w(\Theta)(\varphi_G(X) - \Theta)^2\right),$$

where the expectation is taken with respect to (X, Θ).

Let $X_1, X_2, \ldots, X_n$ be the past data, that are independently and indentically distributed random variables with probability density function $f(x)$. We denote by $\underline{X}_n = (X_1, X_2, \ldots, X_n)$ and $\varphi_n(X) = \varphi_n(X, \underline{X}_n)$ the empirical Bayes estimator of the parameter θ based on past data $\underline{X}_n$ and the present observation X.

Definition 2.1. The conditional Bayes risk of φ_n given $\underline{X}_n$ is

$$R(G, \varphi_n|\underline{X}_n) = E\left(w(\Theta)(\varphi_n(X) - \Theta)^2|\underline{X}_n\right)$$

and the overall Bayes risk of φ_n is

$$R(G, \varphi_n) = E(R(G, \varphi_n|\underline{X}_n)).$$

Here the expectation is taken with respect to $\underline{X}_n$.

Because φ_G is the Bayes estimator, we have

$$R(G, \varphi_G) \le R(G, \varphi_n|\underline{X}_n)$$

$\forall \underline{X}_n$ vector of past data and $\forall n \in \mathbf{N}^*$. Moreover,

$$R(G, \varphi_G) \le R(G, \varphi_n), \quad \forall n \in \mathbf{N}^*. \tag{2.2}$$

Definition 2.2. The nonnegative difference $R(G, \varphi_n) - R(G, \varphi_G)$ is a measure of performance of the empirical Bayes estimator φ_n.

We recall (see [3]) that a sequence of empirical Bayes estimators $(\varphi_n)_{n \ge 1}$ is said to be asymptotically optimal if

$$R(G, \varphi_n) - R(G, \varphi_G) \underset{n \to \infty}{\longrightarrow} 0.$$

Moreover, if $R(G,\varphi_n)-R(G,\varphi_G)=\mathcal{O}(\alpha_n)$, where $(\alpha_n)_{n\geq 1}$ is a sequence of real numbers $\alpha_n>0$ and $\alpha_n \underset{n\to\infty}{\longrightarrow} 0$, then $(\varphi_n)_{n\geq 1}$ is said to be asymptotically optimal with convergence rate of order α_n.

In [4] there were proved Theorems 2.1 and 2.2 below.

THEOREM 2.1. *Under assumption* (A1)–(A5)*, the Bayes estimator of the scale parameter for Pareto distribution is given by*

$$\varphi_G(x)=\begin{cases} x\widetilde{w}(x)-\frac{\widetilde{M}(x)}{x^{\alpha+1}f(x)} & \text{if } 0<x\leq m,\\ m\widetilde{w}(m)-\frac{\widetilde{M}(m)}{m^{\alpha+1}f(m)} & \text{if } x>m,\end{cases} \tag{2.3}$$

where $\widetilde{w}(x)=\frac{w(x)}{q(x)}$, $\widetilde{M}(x)=\frac{M(x)}{q(x)}$ *and* $M(x)=\int_0^x \theta^{\alpha+1}\left(w(\theta)+\theta w'(\theta)\right)\mathrm{d}F(\theta)$.

If $(b_n)_{n\geq 1}$ is a sequence of strictly positive numbers satisfying $b_n \underset{n\to\infty}{\longrightarrow} 0$ and $nb_n \underset{n\to\infty}{\longrightarrow} \infty$, we define

$$f_n(x)=\frac{F_n(x+b_n)-F_n(x)}{b_n},$$

where $F_n(x)$ is the empirical distribution function based on $X_1,X_2,\ldots,X_n$. We note that $f_n(x)$ can be expressed as

$$f_n(x)=\frac{1}{nb_n}\sum_{j=1}^{n} I_{(x,x+b_n]}(X_j). \tag{2.4}$$

Moreover, $E(f_n(x)) \underset{n\to\infty}{\longrightarrow} f(x)$. Thus, $f_n(x)$ is a consistent estimator of $f(x)$ (see[1]).

In [4], there is constructed a consistent estimator of $M(x)$, namely,

$$M_n(x)=\frac{1}{n}\sum_{j=1}^{n} X_j^{\alpha+1}\left(w(X_j)+X_j w'(X_j)\right) I_{(0,x)}(X_j). \tag{2.5}$$

Next the empirical Bayes estimator for the scale parameter θ proposed in [4] is given by

$$\varphi_n(X)=\left[\left(X\widetilde{w}(X)-\frac{\widetilde{M}_n(X)}{X^{\alpha+1}f_n(X)}\right) I_{(0,m]}(X)\vee 0\right]+ \\ +\left[\left(m\widetilde{w}(m)-\frac{\widetilde{M}_n(m)}{m^{\alpha+1}f_n(m)}\right) I_{(m,\infty)}(X)\vee 0\right], \tag{2.6}$$

where $\widetilde{M}_n=\frac{M_n}{q}$ and $a\vee b=\max(a,b)$.

The asymptotic optimality of empirical Bayes estimator (2.6) is asserted by the result below.

THEOREM 2.2. *If* $(b_n)_{n\geq 1}$ *is a sequence of strictly positive numbers satisfying* $b_n \underset{n\to\infty}{\longrightarrow} 0$ *and* $nb_n \underset{n\to\infty}{\longrightarrow} \infty$, $(\varphi_n)_n$ *is the sequence of empirical Bayes estimators* (2.6) *and* φ_G *is the Bayes estimator* (2.3), *then*

$$R(G,\varphi_n) - R(G,\varphi_G) = \mathcal{O}\left(\frac{1}{n}\right) + \mathcal{O}\left(\frac{1}{nb_n}\right) + \mathcal{O}\left(b_n^2\right).$$

3. A LOWER BOUND FOR $R(G,\varphi_n) - R(G,\varphi_G)$ WHEN G IS A UNIFORM DISTRIBUTION FUNCTION

We suppose that $\alpha > 1$, $m = 1$ and G is the uniform on $(0,1)$ cumulative distribution function. Therefore, $g(\theta) = 1$ for $\theta \in (0,1)$. Then

$$f(x) = \begin{cases} \dfrac{\alpha}{\alpha+1} & \text{if } 0 < x \leq 1 \\ \dfrac{\alpha}{\alpha+1}\cdot\dfrac{1}{x^{\alpha+1}} & \text{if } x > 1. \end{cases}$$

It is clear that f is decreasing on $(0,\infty)$ and conditions (A1) and (A2) are fulfilled.

Remark 3.1. For $x < 1$, with the notation from Section 2 we have $E\left(f_n(x)\right) = f(x)$.

We will prove that $R(G,\varphi_n) - R\left(G,\varphi_G\right) \geq \mathcal{O}\left(\frac{1}{nb_n}\right) + \mathcal{O}\left(b_n^2\right)$. In order to do this we need some preliminary results which will be proved following the lines in Liang [2].

First, we take $\delta \in \left(0, \frac{1}{4}\right)$ and for each $x \in (0, 1-\delta]$ we define

$$B_n(x) = I\left(\frac{\widetilde{M}_n(x)}{x^{\alpha+1}f_n(x)} \leq x\widetilde{w}(x)\right)$$

and

$$B_n^c(x) = I\left(\frac{\widetilde{M}_n(x)}{x^{\alpha+1}f_n(x)} > x\widetilde{w}(x)\right).$$

LEMMA 3.1. *We have* $\lim\limits_{n\to\infty} E\left(B_n(x)\right) = 1$ *and* $\lim\limits_{n\to\infty} E\left(B_n^c(x)\right) = 0$.

Proof. We consider only the case where n is sufficiently large, such that $b_n < \delta$. Note that for $x \in (0, 1-\delta]$ we have $E\left(f_n(x)\right) = f(x)$ and then

$$E\left(\widetilde{M}_n(x) - x^{\alpha+2}\widetilde{w}\left(x\right)f_n(x)\right) = \widetilde{M}(x) - x^{\alpha+2}\widetilde{w}(x)f(x),$$

that we suppose to be finite and different from zero. Next, we evaluate

$$\operatorname{Var}\left(\widetilde{M}_n(x) - x^{\alpha+2}\widetilde{w}\,(x)\,f_n(x)\right) \leq$$
$$\leq \frac{1}{n}\frac{1}{q^2(x)}\operatorname{Var}\left(X_j^{\alpha+1}\left(w(X_j)+X_jw'(X_j)\right)I_{(0,x)}(X_j)\right)+$$
$$+\frac{1}{n}\frac{x^{2(\alpha+2)}\widetilde{w}^2\,(x)}{b_n^2}\operatorname{Var}\left(I_{(x,x+b_n]}\,(X_j)\right).$$

Because

$$\operatorname{Var}\left(X_j^{\alpha+1}\left(w(X_j)+X_jw'(X_j)\right)I_{(0,x)}(X_j)\right)\leq c_2^2$$

and

$$\operatorname{Var}\left(I_{(x,x+b_n]}(X_j)\right) = \int_x^{x+b_n}\frac{\alpha}{\alpha+1}\mathrm{d}y - \left(\int_x^{x+b_n}\frac{\alpha}{\alpha+1}\mathrm{d}y\right)^2 =$$
$$= b_nf(x)\left(1-f(x)b_n\right),$$

we have

$$\operatorname{Var}\left(\widetilde{M}_n(x) - x^{\alpha+2}\widetilde{w}\,(x)\,f_n(x)\right) \leq \frac{2}{n}\frac{c_2^2}{q^2(x)} + \frac{2x^{2(\alpha+2)}\widetilde{w}^2(x)f(x)}{nb_n}\underset{n\to\infty}{\longrightarrow}0.$$

Furthermore, the Chebychev inequality ([1]) yields

$$E\left(B_n^c(x)\right) = \Pr\left(\frac{\widetilde{M}_n(x)}{x^{\alpha+1}f_n(x)} > x\widetilde{w}(x)\right) = \Pr\left\{\widetilde{M}_n(x) - x^{\alpha+2}f_n\,(x)\,\widetilde{w}(x) - \right.$$
$$\left. -\left(\widetilde{M}(x) - x^{\alpha+2}f(x)\widetilde{w}(x)\right) > -\left(\widetilde{M}(x) - x^{\alpha+2}f(x)\widetilde{w}(x)\right)\right\} \leq$$
$$\leq \frac{\operatorname{Var}\left(\widetilde{M}_n(x) - x^{\alpha+2}f_n(x)\widetilde{w}(x)\right)}{\left(\widetilde{M}(x) - x^{\alpha+2}\widetilde{w}(x)\frac{\alpha}{\alpha+1}\right)^2}\underset{n\to\infty}{\longrightarrow}0.$$

Then, $\lim\limits_{n\to\infty}E\left(B_n^c\,(x)\right) = 0$ and $\lim\limits_{n\to\infty}E\left(B_n(x)\right) = 1$. □

LEMMA 3.2. *We have* $\int_0^1 E\left(q(x)\left(\varphi_n(x)-\varphi_G(x)\right)^2\right)f\,(x)\,\mathrm{d}x \geq \mathcal{O}\left(\frac{1}{nb_n}\right)$.

Proof. For $x\in(0,1)$ we have $\varphi_G(x) = x\widetilde{w}(x) - \frac{\widetilde{M}(x)}{x^{\alpha+1}f(x)} = x\frac{w(x)}{q(x)} - \frac{M(x)}{q(x)\cdot x^{\alpha+1}f(x)}$ and $\varphi_n(x) = x\widetilde{w}(x) - \frac{\widetilde{M}_n(x)}{x^{\alpha+1}f_n(x)} = x\frac{w(x)}{q(x)} - \frac{M_n(x)}{q(x)\cdot x^{\alpha+1}f_n(x)}$. Then

$$\varphi_n(x) - \varphi_G(x) = \frac{1}{q\,(x)}\left[\frac{M_n(x)}{x^{\alpha+1}f_n\,(x)} - \frac{M(x)}{x^{\alpha+1}f(x)}\right].$$

As in the proof of Theorem 4.1 from [4], we have

$$R(G,\varphi_n)-R(G,\varphi_G)=\int_0^1 E\left(q(x)\left(\varphi_n\left(x\right)-\varphi_G(x)\right)^2\right)f(x)\mathrm{d}x+$$
$$+E\left(q(1)\left(\varphi_n(1)-\varphi_G(1)\right)^2\right)\cdot\left(1-F(1)\right).$$

We see that

$$\int_0^1 E\left(q(x)\left(\varphi_n\left(x\right)-\varphi_G(x)\right)^2\right)f(x)\mathrm{d}x\geq$$
$$\geq\int_0^{1-\delta}E\left(q(x)\left(\varphi_n(x)-\varphi_G(x)\right)^2B_n\left(x\right)\right)f(x)\mathrm{d}x,$$

where

$$q(x)\left(\varphi_n(x)-\varphi_G\left(x\right)\right)^2B_n(x)=\frac{1}{q(x)}\left(\frac{M_n(x)}{x^{\alpha+1}f_n(x)}-\frac{M(x)}{x^{\alpha+1}f(x)}\right)^2B_n^2(x)=$$
$$=\frac{1}{q(x)}\left(\frac{\alpha\left(M_n(x)-M(x)\right)-(\alpha+1)\,M(x)\left(f_n(x)-f(x)\right)}{\alpha x^{\alpha+1}f_n(x)}\right)^2B_n^2(x).$$

We know that $E\left(f_n(x)\right)=f(x)$, $\operatorname{Var}\left(f_n(x)\right)=\frac{1}{nb_n}f(x)\left(1-f(x)b_n\right)$ and $\lim\limits_{n\to\infty}b_n=0$. Therefore, by the Central Limit Theorem ([1]), we get

$$\frac{(\alpha+1)\,M(x)}{\sqrt{q(x)}}\sqrt{nb_n}\left(f_n(x)-f(x)\right)\xrightarrow[n\to\infty]{d}N\left(0,\frac{(\alpha+1)^2M^2(x)}{q(x)}f(x)\right).$$

Moreover,

$$\frac{\alpha\sqrt{nb_n}}{\sqrt{q(x)}}\left(M_n\left(x\right)-M(x)\right)\xrightarrow[n\to\infty]{P}0,\quad \alpha x^{\alpha+1}f_n(x)\xrightarrow[n\to\infty]{P}\alpha x^{\alpha+1}f(x),$$

and

$$\lim_{n\to\infty}E\left(B_n(x)\right)=1.$$

It follows from the Slutsky theorem that

$$\sqrt{nb_n}\left(\frac{\frac{\alpha}{\sqrt{q(x)}}\left(M_n(x)-M(x)\right)-\frac{(\alpha+1)M(x)}{\sqrt{q(x)}}\left(f_n\left(x\right)-f(x)\right)}{\alpha x^{\alpha+1}f_n\left(x\right)}\right)B_n(x)\xrightarrow[n\to\infty]{d}$$
$$\xrightarrow[n\to\infty]{d}N\left(0,\left(\frac{\alpha+1}{\alpha}\right)^2\frac{M^2(x)}{x^{2\alpha+2}q(x)f(x)}\right).$$

Now, we get

$$\varliminf_{n\to\infty}E\left(nb_nq\left(x\right)\left(\varphi_n(x)-\varphi_G(x)\right)^2B_n(x)\right)\geq\left(\frac{\alpha+1}{\alpha}\right)^2\frac{M^2(x)}{x^{2\alpha+2}q(x)f(x)},$$

for all $x \in (0, 1-\delta]$ and $b_n < \delta$. By Fatou's lemma we obtain

$$\varliminf_{n\to\infty} \int_0^{1-\delta} E\left(nb_n q(x)\left(\varphi_n(x) - \varphi_G(x)\right)^2 B_n(x)\right) f(x)\,\mathrm{d}x \geq$$
$$\geq \int_0^{1-\delta} \varliminf_{n\to\infty} E\left(nb_n q(x)\left(\varphi_n(x) - \varphi_G(x)\right)^2 B_n(x)\right) f(x)\mathrm{d}x \geq$$
$$\geq \int_0^{1-\delta} \left(\frac{M(x)}{x^{\alpha+1} f(x)}\right)^2 \frac{1}{q(x)}\mathrm{d}x \geq \frac{\varepsilon^2}{c_1(\alpha+2)^2}\frac{(1-\delta)^3}{3}.$$

Finally, we have

$$\int_0^1 E\left(q(x)\left(\varphi_n(x) - \varphi_G(x)\right)^2\right) f(x)\mathrm{d}x \geq \mathcal{O}\left(\frac{1}{nb_n}\right). \quad \square$$

LEMMA 3.3. *We have* $E\left(q(1)\left(\varphi_n(1) - \varphi_G(1)\right)^2\right) \geq \mathcal{O}\left(b_n^2\right)$ *for* $\alpha \geq 1$.

Proof. Denote

$$A_n = \left\{\sqrt{q(1)}\left|\varphi_n(1) - \varphi_G(1)\right| \geq c_n\right\},$$

where $c_n = \frac{M(1)}{\sqrt{q(1)}f(1)} b_n$. Then

$$E\left(q(1)\left(\varphi_n(1) - \varphi_G(1)\right)^2\right) = E\left(q(1)\left(\varphi_n(1) - \varphi_G(1)\right)^2 B_n(1)\right) +$$
$$+ E\left(q(1)\left(\varphi_n(1) - \varphi_G(1)\right)^2 B_n^c(1)\right).$$

We have

$$E\left(q(1)\left(\varphi_n(1) - \varphi_G(1)\right)^2 B_n^c(1)\right) = q(1)\,\varphi_G^2(1)\Pr\left(B_n^c(1)\right).$$

Furthermore,

$$E\left(q(1)\left(\varphi_n(1) - \varphi_G(1)\right)^2 B_n(1)\right) \geq c_n^2 \Pr\left(A_n \cap B_n(1)\right) \geq$$
$$\geq c_n^2 \Pr\left(\left\{M_n(1) - f_n(1)\left(\frac{M(1)}{f(1)} + c_n\sqrt{q(1)}\right) \geq 0\right\} \cap B_n(1)\right).$$

Thus,

$$E\left(q(1)\left(\varphi_n(1) - \varphi_G(1)\right)^2\right) \geq$$
$$\geq c_n^2 \Pr\left(\left\{M_n(1) - f_n(1)\left(\frac{M(1)}{f(1)} + c_n\sqrt{q(1)}\right) \geq 0\right\} \cap B_n(1)\right) +$$
$$+ q(1)\varphi_G^2(1)\Pr\left(B_n^c(1)\right) \geq$$
$$\geq c_n^2 \Pr\left(\left\{M_n(1) - f_n(1)\left(\frac{M(1)}{f(1)} + c_n\sqrt{q(1)}\right) \geq 0\right\}\right),$$

since for n sufficiently large we have $c_n^2 \leq q(1)\varphi_G^2(1)$ because $c_n = \frac{M(1)}{\sqrt{q(1)}f(1)} b_n \underset{n\to\infty}{\longrightarrow} 0$. Now, we get

$$E\left(q(1)\left(\varphi_n(1) - \varphi_G(1)\right)^2\right) \geq$$
$$\geq c_n^2 \Pr\left\{M_n(1) - f_n(1)\left(\frac{M(1)}{f(1)} + c_n\sqrt{q(1)}\right) \geq 0\right\} =$$
$$= c_n^2 \Pr\left\{M_n(1) - f_n(1)\left(\frac{M(1)}{f(1)} + c_n\sqrt{q(1)}\right) -\right.$$
$$-E\left(M_n(1) - f_n(1)\left(\frac{M(1)}{f(1)} + c_n\sqrt{q(1)}\right)\right) \geq$$
$$\left.\geq -E\left(M_n(1) - f_n(1)\left(\frac{M(1)}{f(1)} + c_n\sqrt{q(1)}\right)\right)\right\}.$$

We also note that

$$E\left(M_n(1) - f_n(1)\left(\frac{M(1)}{f(1)} + c_n\sqrt{q(1)}\right)\right) =$$
$$= M(1)\left[1 - \frac{1}{b_n}\int_1^{1+b_n} \frac{1}{y^{\alpha+1}}\mathrm{d}y - \frac{c_n\sqrt{q(1)}f(1)}{b_n \cdot M(1)}\int_1^{1+b_n} \frac{1}{y^{\alpha+1}}\right] \geq$$
$$\geq M(1)\left[1 - \frac{1}{b_n}\int_1^{1+b_n} \frac{1}{y^2}\mathrm{d}y - \int_1^{1+b_n} \frac{1}{y^2}\right] = 0.$$

Hence, by the last relation and the Central Limit Theorem ([1]),

$$\Pr\left\{M_n(1) - f_n(1)\left(\frac{M(1)}{f(1)} + c_n\sqrt{q(1)}\right) \geq 0\right\} \geq$$
$$\geq \Pr\left\{M_n(1) - f_n(1)\left(\frac{M(1)}{f(1)} + c_n\sqrt{q(1)}\right) -\right.$$
$$\left.-E\left(M_n(1) - f_n(1)\left(\frac{M(1)}{f(1)} + c_n\sqrt{q(1)}\right)\right) \geq 0\right\} \underset{n\to\infty}{\longrightarrow} \frac{1}{2}.$$

Now, we can conclude that

$$E\left(q(1)\left(\varphi_n(1) - \varphi_G(1)\right)^2\right) \geq \mathcal{O}\left(c_n^2\right) = \mathcal{O}\left(b_n^2\right). \quad \square$$

The above results allow one to establish the main result of this paper.

Theorem 3.1. *We have* $R(G,\varphi_n) - R(G,\varphi_G) \geq \mathcal{O}\left(\frac{1}{nb_n}\right) + \mathcal{O}\left(b_n^2\right).$

Proof. By Theorem 4.1 in [4] and Lemmas 3.1–3.3, we have

$$R(G,\varphi_n) - R(G,\varphi_G) = \int_0^1 E\left(q(x)\left(\varphi_n\left(x\right) - \varphi_G(x)\right)^2\right) f(x)\mathrm{d}x +$$

$$+E\left(q(1)\left(\varphi_n(1) - \varphi_G(1)\right)^2\right) \cdot (1 - F(1)) \geq \mathcal{O}\left(\frac{1}{nb_n}\right) + \mathcal{O}\left(b_n^2\right). \quad \square$$

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