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Stratified institutions and elementary homomorphisms

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Abstract

For conventional logic institutions, when one extends the sentences to contain open sentences, their satisfaction is then parameterized. For instance, in the first-order logic, the satisfaction is parameterized by the valuation of unbound variables, while in modal logics it is further by possible worlds. This paper proposes a uniform treatment of such parameterization of the satisfaction relation within the abstract setting of logics as institutions, by defining the new notion of stratified institutions. In this new framework, the notion of elementary model homomorphisms is defined independently of an internal stratification or elementary diagrams. At this level of abstraction, a general Tarski style study of connectives is developed. This is an abstract unified approach to the usual Boolean connectives, to quantifiers, and to modal connectives. A general theorem subsuming Tarski's elementary chain theorem is then proved for stratified institutions with this new notion of connectives.

Keywords: Institutions; Elementary homomorphisms; Elementary colimit theorem; Formal methods

1. Introduction and preliminaries

The theory of institutions [11] is a categorical abstract model theory which formalizes the intuitive notion of logical system, including syntax, semantics, and the satisfaction between them. This emerged in computing science studies of software specification and semantics, in the context of the population explosion of logics there, with the ambition of doing as much as possible at the level of abstraction independent of commitment to any particular logic. Now institutions have become a common tool in the area of algebraic specification, in fact its most fundamental mathematical structure. An *institution* $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ consists of

(i) a category $\mathbb{S}ig$, whose objects are called *signatures*,
(ii) a functor $\mathsf{Sen} : \mathbb{S}ig \to \mathbb{S}et$, giving for each signature a set whose elements are called *sentences* over that signature,
(iii) a functor $\mathsf{Mod} : \mathbb{S}ig^{\mathrm{op}} \to \mathbb{C}at$ giving for each signature Σ a category whose objects are called Σ-*models*, and whose arrows are called Σ-(*model*) *homomorphisms*, and
(iv) a relation $\models_\Sigma \subseteq |\mathsf{Mod}(\Sigma)| \times \mathsf{Sen}(\Sigma)$ for each $\Sigma \in |\mathbb{S}ig|$, called Σ-*satisfaction*,

such that for each morphism $\varphi : \Sigma \to \Sigma'$ in $\mathbb{S}ig$, the *satisfaction condition*

$$M' \models_{\Sigma'} \mathsf{Sen}(\varphi)(e) \quad \text{iff} \quad \mathsf{Mod}(\varphi)(M') \models_\Sigma e$$

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holds for each $M' \in |\mathsf{Mod}(\Sigma')|$ and $e \in \mathsf{Sen}(\Sigma)$. When $M = \mathsf{Mod}(\varphi)(M')$ we will say that M' is an *expansion of M along* φ.

Example 1.1. Let FOL be the institution of *many sorted first-order logic with equality*. Its signatures (S, F, P) consist of a set of sort symbols S, a set F of function symbols, and a set P of relation symbols. Each function or relation symbol comes with a string of argument sorts, called *arity*, and for function symbols, a result sort. Signature morphisms map the three components in a compatible way. Models M are first-order structures interpreting each sort symbol s as a set M_s, each function symbol σ as a function M_σ from the product of the interpretations of the argument sorts to the interpretation of the result sort, and each relation symbol π as a subset M_π of the product of the interpretations of the argument sorts. Notice that each term t of (S, F, P) can be interpreted in any model M as one of its elements, denoted M_t. If $t = \sigma(t_1, \ldots, t_n)$ then M_t is defined as $M_\sigma(M_{t_1}, \ldots, M_{t_n})$. Sentences are the usual first order sentences built from equational and relational atoms by iterative application of Boolean connectives ($\wedge, \neg, \vee, \Rightarrow$, etc.) and quantifiers. Sentence translations rename the sorts, function, and relation symbols. For each signature morphism φ, the reduct $\mathsf{Mod}(\varphi)(M')$ of a model M' is defined by $\mathsf{Mod}(\varphi)(M')_x = M'_{\varphi(x)}$ for each x sort, function, or relation symbol from the domain signature of φ. The satisfaction of sentences by models is the usual Tarskian satisfaction defined inductively on the structure of the sentences.

Example 1.2. Let MPL be the institution of propositional modal logic. The category of signatures is $\mathbb{S}et$, the category of sets and functions. For each set P, the P-sentences are formed from the elements of P by closing under Boolean connectives and unary modal connectives $\Box$ (necessity) and $\Diamond$ (possibility). An MPL model (I, W, R) for a signature P, called *Kripke P-model* consists of

- an index set I,
- a family $W = \{W^i\}_{i \in I}$ of 'possible worlds', which are functions $P \to \{0, 1\}$ (or equivalents subsets of P),
- an 'accessibility' relation $R \subseteq I \times I$.

A *model homomorphism* $h : (I, W, R) \to (I', W', R')$ consists of a function $h : I \to I'$ which preserves the accessibility relation, i.e., $\langle i, j\rangle \in R$ implies $\langle h(i), h(j)\rangle \in R'$, and such that $W^i \subseteq W'^{h(i)}$ for each $i \in I$.

The satisfaction of P-sentences by the Kripke P-models, $(I, W, R) \models \rho$ is defined by $(I, W, R) \models^i \rho$ for each $i \in I$, where $\models^i$ is defined by induction on the structure of the sentences as follows:

- $(I, W, R) \models^i \rho$ iff $\rho \in W^i$ for each $\rho \in P$,
- $(I, W, R) \models^i \rho_1 \wedge \rho_2$ iff $(I, W, R) \models^i \rho_1$ and $(I, W, R) \models^i \rho_2$; and similarly for the other Boolean connectives,
- $(I, W, R) \models^i \Box\rho$ iff $(I, W, R) \models^j \rho$ for each j such that $\langle i, j\rangle \in R$, and
- $\Diamond\rho$ is the same as $\neg\Box\neg\rho$.

A brief random list of examples of institutions in use in computing science include higher-order [3], polymorphic [19], temporal [9], process [9], behavioral [2], coalgebraic [5], object-oriented [12], and multi-algebraic (non-determinism) [17] logics.

In the case of conventional logic institutions, such as FOL, when one extends the sentences to contain open sentences also, the satisfaction of an open sentence is then parameterized by the valuations of the unbound variables. This is in fact how classical approaches introduce the semantic concept of satisfaction. On the other hand, a similar situation arises from the direction of Kripke semantics. In MPL the satisfaction relation is already parameterized by the possible worlds.

This work constitutes an institutional general unified study of such parameterization of the satisfaction relation (between models and sentences) by introducing the concept of 'stratified' institution. These are institutions for which the satisfaction relation is 'stratified' (or parameterized in other words) by 'states of models'. These 'states of models' may be explicit valuations of variables (like in FOL), or implicit possible worlds (like in MPL), or combination of both (like in first order modal logic), or behavioral context (like in hidden algebra [8,12,13,16]), or something else. We show how we can extract canonically an institution out of a 'stratified' institution. At this level we also develop a general Tarski style study of connectives which is an abstract unified approach to the usual Boolean connectives, to quantifiers, and to modal connectives, and we show that this determines canonically a stratified institution (and hence an institution). This way to explicitly structure the satisfaction relation opens the possibility to an institution-independent framework in which various modal and non-modal logics can be treated uniformly. We illustrate this by developing a general concept of elementary (model) homomorphism and by proving a general version of Tarski Elementary Chain Theorem [4,22].

Let us now recall some basic institution-independent concepts which will be used in our paper.

Model amalgamation. This is one of the central properties in institution theory intensely used in application to specification and to model theory. A commuting square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

is an *amalgamation square* if and only if for each Σ_1-model M_1 and a Σ_2-model M_2 such that $\mathsf{Mod}(\varphi_1)(M_1) = \mathsf{Mod}(\varphi_2)(M_2)$, there exists an unique Σ'-model M', denoted $M_1 \otimes M_2$, such that $\mathsf{Mod}(\theta_1)(M') = M_1$ and $\mathsf{Mod}(\theta_2)(M') = M_2$. When dropping the uniqueness condition, we say this is a *weak* amalgamation square.

It is common in actual institutions that all pushout squares of signature morphisms are weak amalgamation squares, in fact most often they are amalgamation squares.

Internal logic. An institution *has conjunctions* [7,21] when for each signature Σ and any Σ-sentences ρ_1 and ρ_2 there exists a Σ-sentence (possibly denoted) $\rho_1 \wedge \rho_2$ such that for each Σ-model M, $M \models \rho_1 \wedge \rho_2$ if and only if $M \models \rho_1$ and $M \models \rho_2$. That an institution 'has' other Boolean connectives may be defined in a similar manner. Notice that while FOL has all Boolean connectives, MPL has only conjunctions.

An institution *has universal $\mathcal{D}$-quantification* [7,21] for a class $\mathcal{D}$ of signature morphisms when for each $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$ and each Σ'-sentence ρ' there exists a Σ-sentence (possibly denoted $(\forall\chi)\rho'$) such that for each Σ-model M, $M \models_\Sigma (\forall\chi)\rho'$ if and only if $M' \models_{\Sigma'} \rho'$ for each Σ'-model M' such that $\mathsf{Mod}(\chi)(M') = M$. Existential quantification can be defined similarly. For example, FOL has universal and existential $\mathcal{D}$-quantification for $\mathcal{D}$ the class of signature extensions with a finite number of constants.

First order quantifications are captured at the institution-independent level by the following weakening of the concept of *representable signature morphism* introduced in [7]. A signature morphism $\chi : \Sigma \to \Sigma'$ is *quasi-representable* [6,15] if and only if each model homomorphism $h : \mathsf{Mod}(\chi)(M') \to N$ has an unique χ-expansion $h' : M' \to N'$. An institution has *quasi-representable $\mathcal{D}$-signature morphisms* for a class $\mathcal{D}$ of signature morphisms when each $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$ is quasi-representable. Note, for example, that FOL has quasi-representable $\mathcal{D}$-signature morphisms for the class $\mathcal{D}$ of signature extensions with constants.

2. Stratified institutions

In any institution $\mathcal{I}$ which has quasi-representable $\mathcal{D}$-signature morphisms, we can define an internal satisfaction of "open sentences" parameterized by abstract "valuations" of internal variables in $\mathcal{D}$ as follows:

Proposition 2.1 *(Internal stratification). Let $\mathcal{I} = (\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, \models)$ be an institution which has quasi-representable $\mathcal{D}$-signature morphisms. Define $St(\mathcal{I}) = (\mathbb{S}ig', \mathsf{Sen}', \mathsf{Mod}', [\![\_]\!])$ as follows:*

- *$\mathbb{S}ig'$ is the category, objects and morphisms of which are every quasi-representable* internal variables (*i.e. quasi-representable signature morphisms*) *$\chi : \Sigma \to \Sigma' \in \mathcal{D}$, and pairs of base institution signature morphisms $\langle\varphi : \Sigma \to \Sigma_1, \varphi' : \Sigma' \to \Sigma'_1\rangle : (\chi : \Sigma \to \Sigma') \to (\chi_1 : \Sigma_1 \to \Sigma'_1)$ such that:*

  $$\begin{array}{ccc} \Sigma & \xrightarrow{\chi} & \Sigma' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ \Sigma_1 & \xrightarrow[\chi_1]{} & \Sigma'_1 \end{array}$$

  is a weak amalgamation square,
- *$\mathsf{Sen}' : \mathbb{S}ig' \to \mathbb{S}et$ is the functor that maps every $\chi : \Sigma \to \Sigma'$ to $\mathsf{Sen}(\Sigma')$,*
- *$\mathsf{Mod}' : \mathbb{S}ig'^{\mathrm{op}} \to \mathbb{C}at$ is the functor that maps $\chi : \Sigma \to \Sigma'$ to $\mathsf{Mod}(\Sigma)$, and*
- *$[\![\_]\!]$ is a $|\mathbb{S}ig'|$-indexed family of functors $[\![\_]\!]_\chi : \mathsf{Mod}'(\chi) \to \mathbb{S}et$ that maps every χ-model M to its* set of states *$[\![M]\!]_\chi = \{M' \in |\mathsf{Mod}(\Sigma')| \mid \mathsf{Mod}(\chi)(M') = M\}$.*

Given $\chi : \Sigma \to \Sigma'$ and a χ-model M, for each state $M' \in [\![M]\!]_\chi$, we define the satisfaction of *$\rho \in \mathsf{Sen}'(\chi)$* by *$M$* at *$M'$, denoted $M \models_\chi^{M'} \rho$, by:*

$$M \models_\chi^{M'} \rho \quad \textit{iff} \quad M' \models_{\Sigma'} \rho.$$

Finally, a χ-model M satisfies *ρ, denoted $M \models_\chi \rho$ if and only if $M \models_\chi^{M'} \rho$ for every $M' \in [\![M]\!]_\chi$.*

Then, $(\mathbb{S}ig', \mathsf{Sen}', \mathsf{Mod}', (\models_\chi)_{\chi \in |\mathbb{S}ig'|})$ is an institution.

Proof. Consider $\langle\varphi, \varphi'\rangle : (\chi : \Sigma \to \Sigma') \to (\chi_1 : \Sigma_1 \to \Sigma'_1)$ a signature morphism in the internal stratification of $\mathcal{I}$, M_1 a χ_1-model and ρ' a χ-sentence. Then $M_1 \models_{\chi_1} \mathsf{Sen}'(\langle\varphi, \varphi'\rangle)(\rho')$ means that $M_1 \models_{\chi_1}^{M'_1} \mathsf{Sen}(\varphi')(\rho')$ for each $M'_1 \in [\![M_1]\!]$ which means $M'_1 \models_{\Sigma'_1} \mathsf{Sen}(\varphi')(\rho')$ for each $M'_1 \in [\![M_1]\!]$, which is equivalent to $\mathsf{Mod}(\varphi')(M'_1) \models_{\Sigma'} \rho'$ for each $M'_1 \in [\![M_1]\!]$.

On the other hand, $\mathsf{Mod}'(\varphi, \varphi')(M_1) \models_\chi \rho'$ means $\mathsf{Mod}(\varphi)(M_1) \models_\chi^{M'} \rho'$ for each $M' \in [\![\mathsf{Mod}(\varphi)(M_1)]\!]$ which means $M' \models_{\Sigma'} \rho'$ for each $M' \in [\![\mathsf{Mod}(\varphi)(M_1)]\!]$.

Now, by the weak amalgamation each $M' \in [\![\mathsf{Mod}(\varphi)(M_1)]\!]$ determines a $M'_1 \in [\![M_1]\!]$. Conversely, each $M'_1 \in [\![M_1]\!]$ determines an $M' \in [\![\mathsf{Mod}(\varphi)(M_1)]\!]$ just by reduction $M' = \mathsf{Mod}(\chi_1)(M'_1)$. This shows that $M_1 \models_{\chi_1} \mathsf{Sen}'(\varphi, \varphi')(\rho')$ iff $\mathsf{Mod}'(\varphi, \varphi')(M_1) \models_\chi \rho'$. □

$St(\mathbb{I})$ is called the *internal stratification* of $\mathbb{I}$.

Hence, internal stratification is a generalization of valuations of unbounded variables.

Example 2.2. In FOL, when we restrict ourselves to the class $\mathcal{D}$ of signature extensions with constants in the role of internal variables χ, we get St(FOL), where signatures are pairs $((S, F, P), X)$ consisting of a FOL-signature and a set of (S-sorted) variables X, and sentences are just "open" (S, F, P)-sentences with unbound (free) variables in X, states of a (S, F, P)-model M are just any valuation $\nu : X \to M$ (hence $[\![M]\!]_\chi = M^X$), and $M \models^{\nu}_{(S,F,P),X} \rho$ means that M satisfies ρ for ν. This is very often how classical logic text books introduce quantifiers in FOL.

$St(\mathcal{I})$ is then the stratification of any institution by explicitly parameterizing the satisfaction condition by valuations of "abstract" variables. However, this is too restrictive when dealing with institutions like modal logics because in this case, $St(\mathcal{I})$ does not take into account parameterization by possible worlds whence the notion of stratified institutions.

Definition 2.3 *(Stratified institution).* A *stratified institution* consists of:

- a category $\mathbb{S}ig$ of signatures,
- a sentence functor $\mathsf{Sen} : \mathbb{S}ig \to \mathbb{S}et$,
- a model functor $\mathsf{Mod} : \mathbb{S}ig^{\mathrm{op}} \to \mathbb{C}at$,
- a "stratification" $[\![\_]\!]$ which consists of a functor $[\![\_]\!]_\Sigma : \mathsf{Mod}(\Sigma) \to \mathbb{S}et$ for each signature $\Sigma \in |\mathbb{S}ig|$ (*states of models*), and a natural transformation $[\![\_]\!]_\varphi : [\![\_]\!]_{\Sigma'} \Rightarrow [\![\_]\!]_\Sigma \circ \mathsf{Mod}(\varphi)$ for each signature morphism $\varphi : \Sigma \to \Sigma'$ such that $[\![M']\!]_\varphi$ is surjective for each $M' \in |\mathsf{Mod}(\Sigma')|$, and
- a satisfaction relation between models and sentences which is parameterized by model states, $M \models^\eta_\Sigma \rho$ where $\eta \in [\![M]\!]_\Sigma$ such that the two following properties are equivalent:
  (i) $\mathsf{Mod}(\varphi)(M) \models_\Sigma^{[\![M]\!]_\varphi(\eta)} \rho$,
  (ii) $M \models^\eta_{\Sigma'} \mathsf{Sen}(\varphi)(\rho)$.

Then, we can define for every $\Sigma \in |\mathbb{S}ig|$, the satisfaction relation $\models_\Sigma \subseteq |\mathsf{Mod}(\Sigma)| \times \mathsf{Sen}(\Sigma)$ as follows: for all $\eta \in [\![M]\!]_\Sigma$

$$M \models_\Sigma \rho \quad \text{if and only if} \quad M \models^\eta_\Sigma \rho.$$

Example 2.4. Given an institution $\mathcal{I}$, $St(\mathcal{I})$ is a stratified institution. Indeed, for each signature morphism $\langle\varphi, \varphi'\rangle : (\chi : \Sigma \to \Sigma') \to (\chi_1 : \Sigma_1 \to \Sigma'_1)$, the natural transformation $[\![\_]\!]_{\langle\varphi,\varphi'\rangle}$ is defined by $[\![M]\!]_{\langle\varphi,\varphi'\rangle}(M') = \mathsf{Mod}(\varphi')(M')$ for each state $M' \in [\![M]\!]_{\chi'}$. The definition of $[\![\_]\!]_\chi$ on model homomorphisms uses the quasi-representable property of χ. The surjectivity of $[\![\_]\!]_{\langle\varphi,\varphi'\rangle}$ is assured by the weak amalgamation property of the square defining $\langle\varphi, \varphi'\rangle$. Notice for St(FOL), the weak amalgamation property corresponding to $\langle\varphi, \varphi'\rangle$ is equivalent to the fact that the mapping $X \to X'$ is injective.

The following is a rather different example of stratified institution where the states of the models are implicit rather than explicit as in the case of the internal stratifications.

Example 2.5. MPL is a stratified institution where $[\![(I, W, R)]\!] = I$ for each set P of propositional variables and each P-model (I, W, R) and for each signature morphism $\varphi : P \to P'$, $[\![(I', W', R')]\!]_\varphi$ is just the identity function on I'.

The following shows that any stratified institution is indeed an institution.

Proposition 2.6. *For each signature morphism $\varphi : \Sigma \to \Sigma'$, each Σ'-model M' and each Σ-sentence ρ, we have*: $M' \models_{\Sigma'} \mathsf{Sen}(\varphi)(\rho)$ *if and only if* $\mathsf{Mod}(\varphi)(M') \models_\Sigma \rho$.

Proof. $M' \models_{\Sigma'} \mathsf{Sen}(\rho)$ iff $M' \models^{\eta'}_{\Sigma'} \rho$ for all states $\eta' \in [\![M']\!]_{\Sigma'}$ iff $\mathsf{Mod}(\varphi)(M') \models_\Sigma^{[\![M']\!]_\varphi(\eta')} \rho$ for all states $\eta' \in [\![M']\!]_{\Sigma'}$. By the surjectivity of $[\![M']\!]_\varphi$, this is equivalent to $\mathsf{Mod}(\varphi)(M') \models^\eta_\Sigma \rho$ for all states $\eta \in [\![\mathsf{Mod}(\varphi)(M')]\!]_\Sigma$, which means $\mathsf{Mod}(\varphi)(M') \models_\Sigma \rho$. □

Notice that the same consequence relation may be parameterized in several ways. For instance, in modal first order logic, one can isolate a pair formed by a possible world and a variable assignment, but also intermediately, just a world, or just an assignment. To compare and relate different stratifications of a same institution, the notions of stratified institution morphisms and comorphisms have been introduced [1]. For lack of space, this will not be presented in this paper.

3. Closed sentences

In FOL a formula without free (unbound) variables has the property that its satisfaction does not depend on valuations of variables. In the literature, such formulæ are called *closed formulæ*. At the level of abstract stratified institutions this concept can be defined in two different ways.

Definition 3.1 *(Semantically closed).* Let $\Sigma \in |\mathbb{S}ig|$ be a signature in a stratified institution. A Σ-sentence ρ is *semantically closed* when for each Σ-model M, the set of states $\{\eta \in [\![M]\!]_\Sigma \mid M \models^\eta_\Sigma \rho\}$ is either $[\![M]\!]_\Sigma$ or $\emptyset$.

(Closed) A signature morphism φ is an institution is *vertical* when $\mathsf{Mod}(\varphi)$ is an identity functor. A Σ-sentence ρ in a stratified institution is *closed* when for each vertical signature morphism $\varphi : \Sigma' \to \Sigma$, there exists a Σ'-sentence $\rho' \in \mathsf{Sen}(\Sigma')$ such that $\mathsf{Sen}(\varphi)(\rho') = \rho$.

Proposition 3.2. *In any internal stratification, each closed sentence is semantically closed. Moreover, if the base institution has all Boolean connectives, quasi-representable $\mathcal{D}$-quantifications, and* Mod *reflects identities then any semantically closed sentence is equivalent to a closed sentence.*

Proof. Let ρ' be a closed $(\chi : \Sigma \to \Sigma')$-sentence. Suppose ρ' is *not* semantically closed. Then there exists a χ-model M for which there are two states $M', M'' \in [\![M]\!]_\chi$ such that $M \models^{M'}_\chi \rho'$ but $M \not\models^{M''}_\chi \rho'$. Hence, we have that $M' \models_{\Sigma'} \rho'$ and $M'' \not\models_{\Sigma'} \rho'$. Consider the vertical signature morphism $\langle 1_\Sigma, \chi\rangle : 1_\Sigma \to \chi$. Therefore, there exists a χ-sentence ρ such that $\mathsf{Sen}'(\langle 1_\Sigma, \chi\rangle)(\rho) = \mathsf{Sen}(\chi)(\rho) = \rho'$. Hence, we have $M' \models_{\Sigma'} \mathsf{Sen}(\chi)(\rho)$ and $M'' \not\models_{\Sigma'} \mathsf{Sen}(\chi)(\rho)$. But, by definition, $M = \mathsf{Mod}(\chi)(M') = \mathsf{Mod}(\chi)(M'')$, and then by the satisfaction condition, we have both $M \models_\Sigma \rho$ and $M \not\models_\Sigma \rho$ what is a contradiction. This shows that ρ' is semantically closed.

For the second part, we first notice that for any internal variable $\chi : \Sigma \to \Sigma' \in \mathcal{D}$ and any Σ-sentence ρ, $\mathsf{Sen}(\chi)(\rho)$ is closed because Mod reflects identities. Then any semantically closed χ-sentence ρ is equivalent to $\mathsf{Sen}(\chi)((\forall\chi)\rho' \vee \neg(\exists\chi)\rho')$. □

While the distinction between closed and semantically closed sentences is not meaningful for FOL, note that in MPL defined as the stratified institution of Example 2.5, each sentence is closed but only propositional logic tautologies or their negations are semantically closed.

From any stratified institution we can "extract" an institution of closed sentences in a canonical way which in the case of actual internal stratifications gives back the base institution. For example, FOL be recovered from its stratification *St*(FOL) by the following steps:

- *St*(FOL)-signatures $((S, F, P), X)$ and $((S, F, P,), X')$ are declared equivalent; their equivalence class corresponds to the FOL-signature (S, F, P),
- the closed $((S, F, P), X)$-sentences are, in fact, just (S, F, P)-sentences in FOL; however, under the above quotienting of signatures, (S, F, P)-sentences in FOL correspond to equivalence classes of closed sentences which are invariant under translations of variables.

The above "extraction" of FOL from *St*(FOL) has been abstractly defined in [1,6] for any stratified institution, but for lack of space, will not be presented in this paper. Interested readers can refer to [6].

4. Elementary homomorphisms

Stratified institutions accommodate a concept of elementary model homomorphisms independently of the existence of an internal stratification or of elementary diagrams (like in [14]). While this captures the usual elementary homomorphisms, it also provides a natural concept of elementary homomorphism for modal logics.

Definition 4.1 *(Elementary homomorphism).* Let Σ be a signature in a stratified institution. A Σ-homomorphism $h : M \to M'$ is *elementary* when for every ρ and every η:

$$M \models^\eta_\Sigma \rho \quad \text{if and only if} \quad M' \models^{[\![h]\!]_\Sigma(\eta)}_\Sigma \rho.$$

Proposition 4.2. *For every stratified institution $\mathbb{S}I$, the three following properties hold*:

(i) *if* $\mathbb{S}I$ *has negation,*[1] *then for every elementary* Σ*-homomorphism* $h : M \to M'$, *M and* M' *are elementary equivalent* (*denoted* $M \equiv_\Sigma M'$), *that is for every semantically closed* Σ*-sentence* ρ, $M \models_\Sigma \rho$ *if and only if* $M' \models_\Sigma \rho$;
(ii) *for every elementary* Σ*-homomorphism* $h : M \to M'$ *such that* $[\![h]\!]_\Sigma$ *is surjective, M and* M' *are elementary equivalent*;
(iii) *for every signature* Σ, *elementary* Σ*-homomorphisms form a subcategory* $Elem(\Sigma)$ *of* $\mathsf{Mod}(\Sigma)$;
(iv) $\mathsf{Mod}(\varphi)$ *preserves elementary homomorphisms.*

Proof. (1), (2), and (3) are straightforward. Let us therefore focus on (4). Let $\varphi : \Sigma \to \Sigma'$ be a signature morphism and let $h : M \to M'$ be an elementary Σ'-homomorphism. Let $\eta \in [\![\mathsf{Mod}(\varphi)(M)]\!]_\Sigma$ and $\rho \in \mathsf{Sen}(\Sigma)$. As $[\![M]\!]_\varphi$ is surjective, there exists $\eta' \in [\![M]\!]_{\Sigma'}$ such that $[\![M]\!]_\varphi(\eta') = \eta$. Therefore, the five following properties are successively equivalent:

- $\mathsf{Mod}(\varphi)(M) \models^{\eta}_{\Sigma} \rho$.
- $M \models^{\eta'}_{\Sigma'} \mathsf{Sen}(\varphi)(\rho)$.
- $M' \models^{[\![h]\!]_{\Sigma'}(\eta')}_{\Sigma'} \mathsf{Sen}(\varphi)(\rho)$.
- $\mathsf{Mod}(\varphi)(M') \models^{[\![M]\!]_\varphi([\![h]\!]_{\Sigma'}(\eta'))}_{\Sigma} \rho$.
- $\mathsf{Mod}(\varphi)(M') \models^{[\![\mathsf{Mod}(\varphi)(h)]\!]_\Sigma(\eta)}_{\Sigma} \rho$.

We then may conclude that $\mathsf{Mod}(\varphi)(h) : \mathsf{Mod}(\varphi)(M) \to \mathsf{Mod}(\varphi)(M')$ is elementary. □

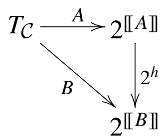
5. Abstract connectives

The definition below provides an abstract notion of connective which generalizes standard Boolean connectives, quantifiers and modalities.

Definition 5.1 *(Connective signature).* A *connective signature* $\mathcal{C}$ is a family $(C_n)_{n\in\mathbb{N}}$ of sets indexed by natural numbers. $c \in C_n$ is called a *connective* of arity *n*. A *morphism* between connective signatures is any $\mathbb{N}$-structure function that preserves arity. By $Log\mathbb{S}ig$ we denote the category of connective signatures.

Notation 5.2. *For any connective signature* $\mathcal{C}$, *let* $T_\mathcal{C}$ *denote the set of all* $\mathcal{C}$*-terms.*

Definition 5.3 *(Connective algebra).* Let $\mathcal{C}$ be a connective signature. A $\mathcal{C}$-algebra *A* consists of a set $[\![A]\!]$, called *carrier* or *set of states*, and a mapping $A : T_\mathcal{C} \to 2^{[\![A]\!]}$. A $\mathcal{C}$-morphism $h : A \to B$ is a mapping $h : [\![A]\!] \to [\![B]\!]$ such that the diagram below commutes:



Let $Log\mathsf{Mod}(\mathcal{C})$ denote the category of $\mathcal{C}$-algebras and let $Log\mathsf{Mod} : Log\mathbb{S}ig \to \mathbb{C}at^{\mathrm{op}}$ be the corresponding functor.

Proposition 5.4. $(Log\mathbb{S}ig, T, Log\mathsf{Mod}, [\![\_]\!], \models)$ *is a stratified institution, where* $T : Log\mathbb{S}ig \to \mathbb{S}et$ *is the functor mapping each connective signature* $\mathcal{C}$ *to* $T_\mathcal{C}$, *and* $A \models^{\eta}_{\mathcal{C}} \rho$ *if and only if* $\eta \in A(\rho)$.

Definition 5.5 *(Connectives).* A stratified institution $(\mathbb{S}ig, \mathsf{Sen}, \mathsf{Mod}, [\![\_]\!], \models)$ *has connectives* if and only if there exists a functor $C : \mathbb{S}ig \to Log\mathbb{S}ig$ and for each $\Sigma \in |\mathbb{S}ig|$ a function

$\beta_\Sigma : |\mathsf{Mod}(\Sigma)| \to |Log\mathsf{Mod}(C(\Sigma))|$,

natural in Σ, such that

- $\mathsf{Sen} = T\_ \circ C$,
- $[\![M]\!]_\Sigma = [\![\beta_\Sigma(M)]\!]_{C(\Sigma)}$ for each Σ-model *M*, and
- $M \models^{\eta}_{\Sigma} \rho$ if and only if $\beta_\Sigma(M) \models^{\eta}_{C(\Sigma)} \rho$.

Example 5.6 *(FOL). St*(FOL) is a stratified institution with connectives in the following manner:

- $C((S, F, P), X)_0 = \{\pi(t) \text{ atom} \mid \pi \in P\}$,[2]
- $C((S, F, P), X)_1 = \{\neg\} \cup \{(\forall x) \mid x \in X\}$,
- $C((S, F, P), X)_2 = \{\wedge\}$, and
- $C((S, F, P), X)_n = \emptyset$ for $n > 2$,
- $[\![M]\!]_{(S,F,P),X} = M^X = \{v \mid v : X \to M\}$,
- $\beta(M)(\pi(t)) = \{v \in M^X \mid v(t) \in M_\pi\}$ for each atom $\pi(t)$,
- $\beta(M)(\neg\rho) = M^X - \beta(M)(\rho)$,
- $\beta(M)(\rho \wedge \rho') = \beta(M)(\rho) \cap \beta(M)(\rho')$,
- $v \in \beta(M)((\forall x)\rho)$ iff $v' \in \beta(M)(\rho)$ for all v' such that $v'(y) = v(y)$ for all $y \neq x$.

Example 5.7 *(MPL).* Modal propositional logic is a stratified institution with connectives in the following manner:

- $C(P)_0 = P$, $C(P)_1 = \{\neg, \diamond\}$, $C(P)_2 = \{\wedge, \vee, \Rightarrow\}$, and $C(P)_n = \emptyset$ for $n > 2$,

[1] $\mathbb{S}I$ has negation when for any sentence ρ there exists a sentence $\neg\rho$ such that $M \models \rho$ iff $M \not\models \neg\rho$.

[2] Here, *t* is a string of terms corresponding to the arity of π.

– $[\![(I, W, R)]\!] = I$,
– $\beta(I, W, R)(\pi) = \{i \mid \pi \in W^i\}$ for each $\pi \in P$,
– for $\neg$ and $\wedge$ the definition is similar to that for *St*(FOL), and we leave the definitions for $\vee$ and $\Rightarrow$ as exercise to the reader,
– $\beta(I, W, R)(\Diamond\rho) = R^{-1}(\beta(I, W, R)(\rho))$.

Notice that internal Boolean connectives such as defined in Section 1 only work for semantically closed sentences. On the contrary, connectives as defined in Definition 5.5 also deal with open sentences.

Proposition 5.8. *For any signature Σ of a stratified institution with connectives, a Σ-homomorphism $h : M \to M'$ is elementary if and only if $[\![h]\!]_\Sigma : [\![M]\!]_\Sigma \to [\![M']\!]_\Sigma$ is a connective algebra morphism $\beta_\Sigma(M) \to \beta_\Sigma(M')$.*

Proof. This follows by the following successive equivalences:

– $h : M \to M'$ elementary,
– $M \models^\eta_\Sigma \rho$ iff $M' \models^{[\![h]\!]_\Sigma(\eta)}_\Sigma \rho$ for all η, ρ,
– $\beta_\Sigma(M) \models^\eta_{C(\Sigma)} \rho$ iff $\beta_\Sigma(M') \models^{[\![h]\!]_\Sigma(\eta)}_{C(\Sigma)} \rho$ for all η, ρ,
– $\eta \in \beta_\Sigma(M)(\rho)$ if and only if $[\![h]\!]_\Sigma(\eta) \in \beta_\Sigma(M')(\rho)$ for all η, ρ,
– $\beta_\Sigma(M)(\rho) = 2^{[\![h]\!]_\Sigma}(\beta_\Sigma(M')(\rho))$. □

Notice that Proposition 5.8 makes β a natural transformation $Elem \Rightarrow Log\mathsf{Mod} \circ C$.

6. Elementary colimit theorem

This section is devoted to the stratified-institution independent generalization of Tarski's elementary chain theorem. Here we provide a general method for proving that colimits of directed diagrams of elementary homomorphisms are still elementary.

Proposition 6.1. *For each connective signature C, $Log\mathsf{Mod}(C)$ has all small colimits.*

Proof. Let $D : J \to Log\mathsf{Mod}(C)$ be a diagram (functor) of C-algebras. Consider the colimit μ of $[\![-]\!]_C \circ D : J \to \mathbb{S}et$.

$$[\![D(i)]\!] \xrightarrow{D(u)} [\![D(j)]\!], \quad [\![D(i)]\!] \xrightarrow{\mu_i} [\![A]\!] \xleftarrow{\mu_j} [\![D(j)]\!]$$

This yields a limit cone

$$2^{[\![D(i)]\!]} \xrightarrow{2^{D(u)}} 2^{[\![D(j)]\!]}, \quad 2^{[\![D(i)]\!]} \xleftarrow{2^{\mu_i}} 2^{[\![A]\!]} \xrightarrow{2^{\mu_j}} 2^{[\![D(j)]\!]}$$

By using the limit property we define $A : T_C \to 2^{[\![A]\!]}$ to be the unique function such that $2^{\mu_i} \circ A = D(i) : T_C \to 2^{[\![D(i)]\!]}$ for each $i \in |J|$. It is easy to check that $(D(i) \xrightarrow{\mu_i} A)_{i \in |J|}$ is the colimit of D. □

Definition 6.2 *(Elementary colimit property).* A stratified institution with connectives has the *elementary colimit property* when for each signature Σ, any colimit $(M_i \xrightarrow{h_i} M)_{i \in |J|}$ of a directed diagram of elementary homomorphisms $(M_i \xrightarrow{h_{i,j}} M_j)_{i<j \in J}$ in $\mathsf{Mod}(\Sigma)$ gets mapped by $[\![-]\!]_\Sigma$ to a colimit $(\beta(M_i) \xrightarrow{[\![h_i]\!]} \beta(M))_{i \in |J|}$ of $(\beta(M_i) \xrightarrow{[\![h_{i,j}]\!]} \beta(M_j))_{i<j \in (J,\leqslant)}$ in $Log\mathsf{Mod}(C(\Sigma))$.

Corollary 6.3. *In any stratified institution with connectives which has the elementary colimit property, the colimit of any directed diagram of elementary homomorphisms consists of elementary homomorphisms too.*

Proof. Immediate from Proposition 5.8. □

Thus in the applications, in order to obtain the conclusion of Corollary 6.3 it is enough to check the elementary colimit property. We resume this section with a couple of examples.

Example 6.4 *(MPL).* Let P be a set and consider a directed colimit of P-models as below (where $(k < j) \in (J, \leqslant)$):

$$(I_k, W_k, R_k) \xrightarrow{h_{k,j}} (I_j, W_j, R_j), \quad (I_k, W_k, R_k) \xrightarrow{h_k} (I, W, R) \xleftarrow{h_j} (I_j, W_j, R_j)$$

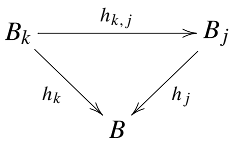
such that all $h_{k,j}$ are elementary. Note that

$$I_k \xrightarrow{h_{k,j}} I_j, \quad I_k \xrightarrow{h_k} I \xleftarrow{h_j} I_j$$

is a colimit in $\mathbb{S}et$ and that W is the unique function $I \to 2^P$ such that $W \circ h_k = W_k$ for each $k \in J$. This definition works because $(W_k)_{k \in J}$ is a co-cone over

$(I_k \xrightarrow{h_{k,j}} I_j)_{(k\leqslant j)\in(J,\leqslant)}$ which is due the fact that each $h_{k,j}$ is elementary. Moreover $R = \bigcup_{k\in J}(h_k \times h_k)(R_k)$.

Now let $B_k = \beta(I_k, W_k, R_k)$ for each $k \in J$ (thus $[\![B_k]\!] = I_k$) and consider the corresponding colimit of connective algebras, cf. Proposition 6.1.



Note that $[\![B]\!] = I$ and that by Proposition 6.1 we have that

$$B_k(\rho) = h_k^{-1}\big(B(\rho)\big) \quad \text{for each } k \in J.$$

In order to establish the elementary colimit property we have to show that $B = \beta(I, W, R) : T_{C(P)} \to 2^{[\![B]\!]} = 2^I$. We do this by induction on the structure of $\rho \in T_{C(P)}$. For this we will make use repeatedly of the fact that for each $X, Y \subseteq I$, $X = Y$ if and only if $h_k^{-1}(X) = h_k^{-1}(Y)$ for each $k \in J$, which is justified by the fact that $(h_k)_{k\in J}$ is a colimit in $\mathbb{S}et$.

For any $\pi \in P$, let us show that $B(\pi) = \beta(I, W, R)(\pi)$. For each $k \in J$, $h_k^{-1}(\beta(I, W, R)(\pi)) = h_k^{-1}\{i \in I \mid \pi \in W^i\} = \{i_k \in I_k \mid \pi \in W^{h_k(i_k)} = W_k^{i_k}\} = B_k(\pi) = h_k^{-1}(B(\pi))$.

For any $\rho, \rho' \in T_{C(P)}$, let us show that $B(\rho \wedge \rho') = B(\rho) \cap B(\rho')$. For each $k \in J$, $h_k^{-1}(B(\rho \wedge \rho')) = B_k(\rho \wedge \rho') = B_k(\rho) \cap B_k(\rho') = h_k^{-1}(B(\rho)) \cap h_k^{-1}(B(\rho')) = h_k^{-1}(B(\rho) \cap B(\rho'))$. The other Boolean connectives can be handled similarly.

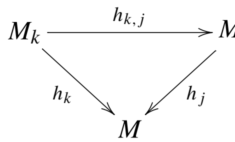
Finally, for any $\rho \in T_{C(P)}$ we show that $B(\Diamond\rho) = R^{-1}(B(\rho))$. For each $k \in J$, $h_k^{-1}(B(\Diamond\rho)) = B_k(\Diamond\rho) = R_k^{-1}(B_k(\rho)) = R_k^{-1}(h_k^{-1}(B(\rho)))$. This should be equal to $h_k^{-1}(R^{-1}(B(\rho)))$. That

$$R_k^{-1}\big(h_k^{-1}\big(B(\rho)\big)\big) \subseteq h_k^{-1}\big(R^{-1}\big(B(\rho)\big)\big)$$

is immediate. Because $(J, \leqslant)$ is directed, the opposite inclusion also holds for some $k' > k$. Hence

$$\begin{aligned} h_k^{-1}\big(B(\Diamond\rho)\big) &= h_{k,k'}^{-1}\big(h_{k'}^{-1} B(\Diamond\rho)\big) \\ &= h_{k,k'}^{-1}\big(h_{k'}^{-1}\big(R^{-1}(B(\rho))\big)\big) \\ &= h_k^{-1}\big(R^{-1}\big(B(\rho)\big)\big). \end{aligned}$$

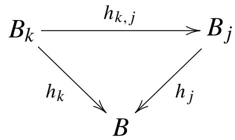
Example 6.5 *(FOL).* We apply the same method as in Example 6.4. Let (S, F, P) be a FOL-signature and X be an S-sorted set of variables. Consider a directed colimit of (S, F, P)-models as below (where $(k < j) \in (J, \leqslant)$):



such that all $h_{k,j}$ are elementary.

At the level of carrier sets, M is the set of equivalence classes of the equivalence relation $\sim$ on the disjoint union $\coprod_{k\in J} M_k$ defined by $m_k \sim m_{k'}$ (for $m_k \in M_k$ and $m_{k'} \in M_{k'}$) if and only if there exists $k, k' < j$ such that $h_{k,j}(m_k) = h_{k',j}(m_{k'})$. Because each $h_{k,j}$ is elementary, it is injective. From this it follows that each h_k is injective too. Note also that for each $\pi \in P$, $M_\pi = \{m_k/{\sim} \mid m_k \in (M_k)_\pi, k \in J\}$.

Now let $B_k = \beta(M_k)$ for each $k \in J$ (thus $[\![B_k]\!] = M_k^X$) and consider the corresponding colimit of connective algebras, cf. Proposition 6.1.



Note that $[\![B]\!] = M^X$. (Here by $h_k^X : M_k^X \to M^X$ we mean the function which maps each $v_k \in M_k^X$ to $h_k \circ v_k$.) By Proposition 6.1 we also have that

$$B_k(\rho) = \big(h_k^X\big)^{-1}\big(B(\rho)\big) \quad \text{for each } k \in J.$$

We have to show that $B = \beta(M) : T_{C((S,F,P),X)} \to 2^{[\![B]\!]} = 2^{(M^X)}$. We do this by induction on the structure of $\rho \in T_{C((S,F,P),X)}$.

For any atom $\pi(t)$, let us show that $B(\pi(t)) = \beta(M)(\pi(t))$. For each $k \in J$, $(h_k^X)^{-1}(\beta(M)(\pi(t))) = (h_k^X)^{-1}\{v \in M^X \mid v(t) \in M_\pi\} = \{v_k \in M_k^X \mid h_k(v_k(t)) \in M_\pi\} = \{v_k \in M_k^X \mid v_k(t) \in (M_k)_\pi\} = B_k(\pi(t)) = (h_k^X)^{-1}(B(\pi(t)))$.

The Boolean connectives are handled exactly like in Example 6.4.

Finally, for any $\rho \in T_{C((S,F,P),X)}$ and any $k \in J$, $(h_k^X)^{-1}(B((\forall x)\rho)) = B_k((\forall x)\rho) = \{v_k \in M_k^X \mid v'_k(y) = v_k(y)$ for all $y \neq x$ implies $v'_k \in B_k(\rho)\}$ and $(h_k^X)^{-1}\beta(M)((\forall x)\rho) = (h_k^X)^{-1}\{v \in M^X \mid v'(y) = v(y)$ for all $y \neq x$ implies $v' \in \beta(M)(\rho) = B(\rho)\} = \{v_k \in M_k^X \mid v'(y) = h_k(v_k(y))$ for all $y \neq x$ implies $v' \in B(\rho)\}$.

For each $v_k \in (h_k^X)^{-1}\beta(M)((\forall x)\rho)$, each v'_k such that $v'_k(y) = v_k(y)$ for all $y \neq x$ determines $v' = h_k \circ v'_k$ with $v'(y) = h_k(v_k(y))$ for all $y \neq x$. Hence $v' \in B(\rho)$ and thus $v'_k \in B_k(\rho) = (h_k^X)^{-1}(B(\rho))$. This means $v_k \in (h_k^X)^{-1}(B((\forall x)\rho))$.

For each $v_k \in (h_k^X)^{-1}B((\forall x)\rho)$, let $v' \in M^X$ such that $v'(y) = h_k(v_k(y))$ for all $y \neq x$. We have to prove that $v' \in B(\rho)$. Because M is a directed co-limit there exists $k' > k$ such that $v'(x) \in h_{k'}(M_{k'})$. Thus there exists $v'_{k'} \in M^X_{k'}$ such that $v' = h_{k'} \circ v'_{k'}$. Let $v_{k'} = h_{k,k'} \circ v_k$. For each $y \neq x$, $h_{k'}(v_{k'}(y)) = h_{k'}(h_{k,k'}(v_k(y))) = h_k(v_k(y)) = v'(y) = h_{k'}(V'_{k'}(y))$. Then $v_{k'}(y) = v'_{k'}(y)$ since $h_{k'}$ is injective. Because $v_{k'} \in B_{k'}((\forall x)\rho)$ we deduce that $v'_{k'} \in B_{k'}(\rho)$ and further that $v' = h_{k'} \circ v'_{k'} \in B(\rho)$.

7. Conclusions

We have introduced stratified institutions and a general theory of connectives for capturing models with 'states' in an uniform institution-independent manner. This unifies the model theory of logics with explicit states of models (e.g., variables valuations in FOL), with implicit states (e.g., modal logics), or with both kinds (e.g., modal first order logic, hidden algebra, etc.). As an application we have studied elementary (model) homomorphisms in this setting. The main result here, given by Corollary 6.3, should be regarded as a method to establish the elementary colimit theorem in institutions, which functions at a meta-level with respect to the institution-independent method proposed in [14]. The latter generalizes proofs corresponding to the level of Example 6.5.

Several issues can be pursued with stratified institutions. First of all, many more results of model theory will have to be developed. Finally, since pioneer works of D. Gabbay [10] and Fiblog group [20], considering abstract forms of connectives have shown their importance for another problem in computing science, namely combining logics. Parchments and its extensions brought a first rather unsatisfactory answer [18]. Stratified institutions promise to bring a more abstract and satisfactory answer to this problem.

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