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What is a Logic?

In memoriam Joseph Goguen

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Abstract. This paper builds on the theory of institutions, a version of abstract model theory that emerged in computer science studies of software specification and semantics. To handle proof theory, our institutions use an extension of traditional categorical logic with sets of sentences as objects instead of single sentences, and with morphisms representing proofs as usual. A natural equivalence relation on institutions is defined such that its equivalence classes are logics. Several invariants are defined for this equivalence, including a Lindenbaum algebra construction, its generalization to a Lindenbaum category construction that includes proofs, and model cardinality spectra; these are used in some examples to show logics inequivalent. Generalizations of familiar results from first order to arbitrary logics are also discussed, including Craig interpolation and Beth definability.

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1. Introduction

Logic is often informally described as the study of *sound reasoning*. As such, it plays a crucial role in several areas of mathematics (especially foundations) and of computer science (especially formal methods), as well as in other fields, such as analytic philosophy and formal linguistics. In an enormous development beginning in the late 19^{th} century, it has been found that a wide variety of different principles are needed for sound reasoning in different domains, and "a logic" has come to mean a set of principles for some form of sound reasoning. But in a subject the essence of which is formalization, it is embarrassing that there is no widely acceptable formal definition of "a logic". It is clear that two key problems here are to define what it means for two presentations of a logic to be equivalent, and to provide effective means to demonstrate equivalence and inequivalence.

This paper addresses these problems using the notion of "institution", which arose within computer science in response to the population explosion among the logics in use there, with the ambition of doing as much as possible at a level of abstraction independent of commitment to any particular logic [19, 36, 21]. The *soundness* aspect of sound reasoning is addressed by axiomatizing the notion of satisfaction, and the *reasoning* aspect is addressed by calling on categorical logic, which applies category theory to proof theory by viewing proofs as morphisms. Thus, institutions provide a balanced approach, in which both syntax and semantics play key roles. However, much of the institutional literature considers sentences without proofs and models without (homo)morphisms, and a great deal can be done just with satisfaction, such as giving general foundations for modularization of specifications and programs, which in turn has inspired aspects of the module systems of programming languages including C++, ML, and Ada.

Richer variants of the institution notion consider entailment relations on sentences and/or morphisms of models, so that they form categories; using proof terms as sentence morphisms provides a richer variant which fully supports proof theory. We call these the set/set, set/cat, cat/set, and cat/cat variants (where the first term refers to sentences, and the second to models); the table in Thm. 5.20 summarizes many of their properties. See [21] for a general treatment of the variant notions of institution, and [38, 40, 11, 12, 14] for some non-trivial results in abstract model theory done institutionally.

This paper adds to the literature on institutions a notion of equivalence, such that a logic is an equivalence class of institutions. To support this thesis, we consider a number of logical properties, model and proof theoretical, that are, and that are not, preserved under equivalence, and apply them to a number of examples. Perhaps the most interesting invariants are versions of the Lindenbaum algebra; some others concern cardinality of models. We also develop a normal form for institutions under our notion of equivalence, by extending the categorical notion of "skeleton".

We extend the Lindenbaum algebra construction to a *Lindenbaum category* construction, defined on any institution with proofs, by identifying not only equivalent sentences, but also equivalent proofs. We show that this construction is an invariant, i.e., preserved up to isomorphism by our equivalence on institutions. This construction extends the usual approach of categorical logic by having sets of sentences as objects, rather than just single sentences, and thus allows treating a much larger class of logics in a uniform way.

A perhaps unfamiliar feature of institutions is that satisfaction is not a dyadic relation, but rather a relation among sentence, model, and "signature", where signatures form a category the objects of which are thought of as vocabularies over which the sentences are constructed. In concrete cases, these may be propositional variables, relation symbols, function symbols, and so on. Since these form a category, it is natural that the constructions of sentences (or formulae) and models appear as functors on this category, and it is also natural to have an axiom expressing the invariance of "truth" (i.e., satisfaction) under change of notation. See Def. 2.1 below. When the vocabulary is fixed, the category of signatures is the one object category **1**. (Another device can be used to eliminate models, giving pure proof theory as a special case, if desired.) If $\sigma\colon \Sigma \longrightarrow \Sigma'$ is an inclusion of signatures, then its application to models (via the model functor) is "reduct." The institutional triadic satisfaction can be motivated philosophically by arguments like those given by Peirce [30] for his "interpretants," which allow for context dependency of denotation in his semiotics, as opposed to Tarski's dyadic satisfaction. We also use this feature to resolve a problem about cardinality raised in [3]; see Example 2.2.

Joseph Goguen. Our co-author Joseph Goguen died on July 3rd, 2006. The scientific community has lost a great scientist doing pioneering research in many diverse areas, and we also have lost a close friend and teacher. Shortly before his death, we were privileged to take part in the Festschrift colloquium for his 65th birthday. His most important message to the Festschrift participants was a commitment to solidarity and cooperation.

2. Institutions and Logics

We assume the reader is familiar with basic notions from category theory; e.g., see [1, 25] for introductions to this subject. By way of notation, $|\mathbb{C}|$ denotes the class of objects of a category $\mathbb{C}$, and composition is denoted by "$\circ$". Categories are assumed by convention to be locally small (i.e., to have a small set of morphisms between any two objects) unless stated otherwise. The basic concept of this paper in its set/cat variant is as follows[1]:

Definition 2.1. An *institution* $\mathcal{I} = (\mathbb{S}ign^{\mathcal{I}}, \mathsf{Sen}^{\mathcal{I}}, \mathsf{Mod}^{\mathcal{I}}, \models^{\mathcal{I}})$ consists of

1. a category $\mathbb{S}ign^{\mathcal{I}}$, whose objects are called *signatures*,
2. a functor $\mathsf{Sen}^{\mathcal{I}} : \mathbb{S}ign^{\mathcal{I}} \to \mathbb{S}et$, giving for each signature a set whose elements are called *sentences* over that signature,
3. a functor $\mathsf{Mod}^{\mathcal{I}} : (\mathbb{S}ign^{\mathcal{I}})^{\mathrm{op}} \to \mathbb{CAT}$ giving for each signature Σ a category whose objects are called Σ-*models*, and whose arrows are called Σ-(*model*) *morphisms*[2] and
4. a relation $\models^{\mathcal{I}}_{\Sigma} \subseteq |\mathsf{Mod}^{\mathcal{I}}(\Sigma)| \times \mathsf{Sen}^{\mathcal{I}}(\Sigma)$ for each $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$, called Σ-*satisfaction*,

[1] A more concrete definition is given in [22], which avoids category theory by spelling out the conditions for functoriality, and assuming a set theoretic construction for signatures. Though less general, this definition is sufficient for everything in this paper; however, it would greatly complicate our exposition. Our use of category theory is modest, oriented towards providing easy proofs for very general results, which is precisely what is needed for the goals of this paper.

[2] $\mathbb{CAT}$ is the category of all categories; strictly speaking, it is only a quasi-category living in a higher set-theoretic universe. See [25] for a discussion of foundations.

such that for each morphism $\sigma : \Sigma \to \Sigma'$ in $\mathbb{S}ign^{\mathcal{I}}$, the *satisfaction condition*

$$M' \models^{\mathcal{I}}_{\Sigma'} \mathsf{Sen}^{\mathcal{I}}(\sigma)(\varphi) \text{ iff } \mathsf{Mod}^{\mathcal{I}}(\sigma)(M') \models^{\mathcal{I}}_{\Sigma} \varphi$$

holds for each $M' \in |\mathsf{Mod}^{\mathcal{I}}(\Sigma')|$ and $\varphi \in \mathsf{Sen}^{\mathcal{I}}(\Sigma)$. We denote the *reduct* functor $\mathsf{Mod}^{\mathcal{I}}(\sigma)$ by $\_\restriction_\sigma$ and the sentence translation $\mathsf{Sen}^{\mathcal{I}}(\sigma)$ by $\sigma(\_)$. When $M = M'\restriction_\sigma$ we say that M' is a *σ-expansion of M*.

A *set/set institution* is an institution where each model category is discrete; this means that the model functor actually becomes a functor $\mathsf{Mod}^{\mathcal{I}} : (\mathbb{S}ign^{\mathcal{I}})^{\mathrm{op}} \to \mathbb{C}lass$ into the quasi-category of classes and functions.
General assumption: We assume that all institutions are such that satisfaction is invariant under model isomorphism, i.e. if Σ-models M, M' are isomorphic, then $M \models_\Sigma \varphi$ iff $M' \models_\Sigma \varphi$ for all Σ-sentences φ. □

We now consider classical propositional logic, perhaps the simplest non-trivial example (see the extensive discussion in [3]), and also introduce some concepts from the theory of institutions:

Example 2.2. Fix a countably infinite[3] set $\mathcal{X}$ of variable symbols, and let $\mathbb{S}ign$ be the category with finite subsets Σ of $\mathcal{X}$ as objects, and with inclusions as morphisms (or all set maps, if preferred, it matters little). Let $\mathsf{Mod}(\Sigma)$ have $[\Sigma \to \{0,1\}]$ (the set of functions from Σ to $\{0,1\}$) as its set of objects; these models are the row labels of truth tables. Let a (unique) Σ-model morphism $h\colon M \longrightarrow M'$ exist iff for all $p \in \Sigma$, $M(p) = 1$ implies $M'(p) = 1$. Let $\mathsf{Mod}(\Sigma' \hookrightarrow \Sigma)$ be the restriction map $[\Sigma \to \{0,1\}] \to [\Sigma' \to \{0,1\}]$. Let $\mathsf{Sen}(\Sigma)$ be the (absolutely) free algebra generated by Σ over the propositional connectives (we soon consider different choices), with $\mathsf{Sen}(\Sigma \hookrightarrow \Sigma')$ the evident inclusion. Finally, let $M \models_\Sigma \varphi$ mean that φ evaluates to true (i.e., 1) under the assignment M. It is easy to verify the satisfaction condition, and to see that φ is a tautology iff $M \models_\Sigma \varphi$ for all $M \in |\mathsf{Mod}(\Sigma)|$. Let **CPL** denote this institution of propositional logic, with the connectives conjunction, disjunction, negation, implication, true and false. Let $\mathbf{CPL}^{\neg,\wedge,false}$ denote propositional logic with negation, conjunction and false, and $\mathbf{CPL}^{\neg,\vee,true}$ with propositional logic negation, disjunction and true[4].

This arrangement puts truth tables on the side of semantics, and formulas on the side of syntax, each where it belongs, instead of trying to treat them the same way. It also solves the problem raised in [3] that the cardinality of $\mathcal{L}(\Sigma)$ varies with that of Σ, where $\mathcal{L}(\Sigma)$ is the quotient of $\mathsf{Sen}(\Sigma)$ by the semantic equivalence $\models\mid_\Sigma$, defined by $\varphi \models\mid_\Sigma \varphi'$ iff ($M \models_\Sigma \varphi$ iff $M \models_\Sigma \varphi'$, for all $M \in |\mathsf{Mod}(\Sigma)|$); it is the Lindenbaum algebra, in this case, the free Boolean algebra over Σ, and its cardinality is 2^{2^n} where n is the cardinality of Σ. Hence this cardinality cannot be considered an invariant of **CPL** without the parameterization by Σ (see also Def. 4.13 below). □

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[3]The definition also works for finite or uncountable $\mathcal{X}$.
[4]The truth constant avoids the empty signature having no sentences at all.

The moral of the above example is that everything should be parameterized by signature. Although the construction of the underlying set of the Lindenbaum algebra above works for any institution, its algebraic structure depends on how sentences are defined. However, Section 4 shows how to obtain at least part of this structure for any institution.

Example 2.3. The institution **FOLR** of unsorted first-order logic with relations has signatures Σ that are families Σ_n of sets of relation symbols of arity $n \in \mathbb{N}$, and **FOLR** signature morphisms $\sigma\colon \Sigma \longrightarrow \Sigma'$ that are families $\sigma_n\colon \Sigma_n \longrightarrow \Sigma'_n$ of arity-preserving functions on relation symbols. An **FOLR** Σ-sentence is a closed first-order formula using relation symbols in Σ, and sentence translation is relation symbol substitution. A **FOLR** Σ-model is a set M and a subset $R_M \subseteq M^n$ for each $R \in \Sigma_n$. Model translation is reduct with relation translation. A Σ-model morphism is a function $h\colon M \longrightarrow M'$ such that $h(R_M) \subseteq R_{M'}$ for all R in Σ. Satisfaction is as usual. The institution **FOL** adds function symbols to **FOLR** in the usual way, and **MSFOL** is its many sorted variant. □

Example 2.4. In the institution **EQ** of many sorted equational logic, a signature consists of a set of sorts with a set of function symbols, each with a string of argument sorts and a result sort. Signature morphisms map sorts and function symbols in a compatible way. Models are many sorted algebras, i.e., each sort is interpreted as a carrier set, and each function symbol names a function among carrier sets specified by its argument and result sorts. Model translation is reduct, sentences are universally quantified equations between terms of the same sort, sentence translation replaces translated symbols (assuming that variables of distinct sorts never coincide in an equation), and satisfaction is the usual satisfaction of an equation in an algebra. □

Example 2.5. **K** is propositional modal logic with $\Box$ and $\Diamond$. Its models are Kripke structures, and satisfaction is defined using possible-world semantics in the usual way. **IPL** is intuitionistic propositional logic, differing from **CPL** in having Kripke structures as models, and possible-world satisfaction. The proof theory of **IPL** (which is favored over the model theory by intuitionists) is discussed in Section 5. □

Both intuitionistic and modal logic in their first-order variants, with both constant and varying domains, form institutions, as do other modal logics restricting **K** by further axioms, such as **S4** or **S5**, as well as substructural logics, like linear logic, where judgements of the form $\varphi_1 \ldots \varphi_n \vdash \psi$ are sentences. Higher-order [7], polymorphic [37], temporal [18], process [18], behavioural [4], coalgebraic [9] and object-oriented [20] logics also form institutions. Many familiar basic concepts can be defined over any institution:

Definition 2.6. Given a set of Σ-sentences Γ and a Σ-sentence φ, then φ is a *semantic consequence* of Γ, written $\Gamma \models_\Sigma \varphi$ iff for all Σ-models M, we have $M \models_\Sigma \Gamma$ implies $M \models_\Sigma \varphi$, where $M \models_\Sigma \Gamma$ means $M \models_\Sigma \psi$ for each $\psi \in \Gamma$. Two sentences are *semantically equivalent*, written $\varphi_1 \models\!\mid \varphi_2$, if they are satisfied by the same models. Two models are *elementary equivalent*, written $M_1 \equiv M_2$, if they satisfy the same sentences. An institution is *compact* iff $\Gamma \models_\Sigma \varphi$ implies $\Gamma' \models_\Sigma \varphi$ for some finite subset Γ' of Γ. A *theory* is a pair (Σ, Γ) where Γ is a set of Σ-sentences, and is *consistent* iff it has at least one model. □

Cardinality properties associate cardinalities to objects in a category. It is natural to do this using *concrete categories* [1], which have a faithful *forgetful* or *carrier functor* to $\mathbb{S}et$. Since we also treat many sorted logics, we generalize from $\mathbb{S}et$ to categories of many sorted sets $\mathbb{S}et^S$, where the sets S range over sort sets of an institution's signatures. The following enriches institutions with carrier sets for models [5]:

Definition 2.7. A *concrete institution* is an institution $\mathcal{I}$ together with a functor $sorts^{\mathcal{I}} : \mathbb{S}ign^{\mathcal{I}} \to \mathbb{S}et$ and a natural transformation $|\_|^{\mathcal{I}} : \mathsf{Mod}^{\mathcal{I}} \to \mathbb{S}et^{(sorts^{\mathcal{I}})^{op}(\_)}$ between functors from $\mathbb{S}ign^{op}$ to $\mathbb{CAT}$ such that for each signature Σ, the *carrier functor* $|\_|^{\mathcal{I}}_{\Sigma} : \mathsf{Mod}^{\mathcal{I}}(\Sigma) \to \mathbb{S}et^{sorts^{\mathcal{I}}(\Sigma)}$ is faithful (that is, $\mathsf{Mod}^{\mathcal{I}}(\Sigma)$ is a concrete category, with carrier functors $|\_|^{\mathcal{I}}_{\Sigma} : \mathsf{Mod}^{\mathcal{I}}(\Sigma) \to \mathbb{S}et^{sorts^{\mathcal{I}}(\Sigma)}$ natural in Σ). Here, $\mathbb{S}et^{(sorts^{\mathcal{I}})^{op}(\_)}$ stands for the functor that maps each signature $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$ to the category of $sorts^{\mathcal{I}}(\Sigma)$-sorted sets. A concrete institution has the *finite model property* if each satisfiable theory has a finite model (i.e., a model M with the carrier $|M|$ being a family of finite sets). A concrete institution *admits free models* if all carrier functors for model categories have left adjoints. □

The following notion from [29] also provides signatures with underlying sets of symbols, by extending $sorts^{\mathcal{I}}$; essentially all institutions that arise in practice have this structure:

Definition 2.8. A *concrete institution with symbols* is a concrete institution $\mathcal{I}$ together with a faithful functor $Symb^{\mathcal{I}} : \mathbb{S}ign^{\mathcal{I}} \to \mathbb{S}et$ that naturally extends $sorts^{\mathcal{I}}$, that is, such that for each signature Σ, $sorts^{I}(\Sigma) \subseteq Symb^{\mathcal{I}}(\Sigma)$, and for each σ in $\mathbb{S}ign^{\mathcal{I}}$, $Symb^{\mathcal{I}}(\sigma)$ extends $sorts^{\mathcal{I}}(\sigma)$. A concrete institution with symbols *admits free models* if all the forgetful functors for model categories have left adjoints. □

3. Equivalence of Institutions

Relationships between institutions are captured mathematically by 'institution morphisms', of which there are several variants, each yielding a category under a canonical composition. For the purposes of this paper, institution comorphisms [21] are technically more convenient, though the definition of institution equivalence below is independent of this choice. The original notion, from [19] in the set/cat form, works well for 'forgetful' morphisms from one institution to another having less structure:

Definition 3.1. Given institutions $\mathcal{I}$ and $\mathcal{J}$, then an *institution morphism* $(\Phi, \alpha, \beta) : \mathcal{I} \to \mathcal{J}$ consists of

1. a functor $\Phi : \mathbb{S}ign^{\mathcal{I}} \to \mathbb{S}ign^{\mathcal{J}}$,

2. a natural transformation $\alpha : \mathsf{Sen}^{\mathcal{J}} \circ \Phi \Rightarrow \mathsf{Sen}^{\mathcal{I}}$, and
3. a natural transformation $\beta : \mathsf{Mod}^{\mathcal{I}} \Rightarrow \mathsf{Mod}^{\mathcal{J}} \circ \Phi^{\mathrm{op}}$

such that the following *satisfaction condition* holds

$$M \models^{\mathcal{I}}_{\Sigma} \alpha_{\Sigma}(\varphi) \;\text{ iff }\; \beta_{\Sigma}(M) \models^{\mathcal{I}}_{\Phi(\Sigma)} \varphi$$

for each signature $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$, each Σ-model M and each $\Phi(\Sigma)$-sentence φ. □

Institution morphisms form a category $\mathbb{I}ns$ under the natural composition.

Definition 3.2. Given institutions $\mathcal{I}$ and $\mathcal{J}$, then an *institution comorphism* $(\Phi, \alpha, \beta)\colon \mathcal{I} \longrightarrow \mathcal{J}$ consists of

- a functor $\Phi\colon \mathbb{S}ign^{\mathcal{I}} \longrightarrow \mathbb{S}ign^{\mathcal{J}}$,
- a natural transformation $\alpha : \mathsf{Sen}^{\mathcal{I}} \Rightarrow \mathsf{Sen}^{\mathcal{J}} \circ \Phi$,
- a natural transformation $\beta : \mathsf{Mod}^{\mathcal{J}} \circ \Phi^{op} \Rightarrow \mathsf{Mod}^{\mathcal{I}}$

such that the following *satisfaction condition* is satisfied for all $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$, $M' \in |\mathsf{Mod}^{\mathcal{J}}(\Phi(\Sigma))|$ and $\varphi \in \mathsf{Sen}^{\mathcal{I}}(\Sigma)$:

$$M' \models^{\mathcal{J}}_{\Phi(\Sigma)} \alpha_{\Sigma}(\varphi) \;\text{ iff }\; \beta_{\Sigma}(M') \models^{\mathcal{I}}_{\Sigma} \varphi \,.$$

With the natural compositions and identities, this gives a category $\mathbb{C}oIns$ of institutions and institution comorphisms.

A *set/set institution comorphism* is like a set/cat comorphism, except that β_{Σ} is just a function on the objects of model categories; the model morphisms are ignored.

Given concrete institutions $\mathcal{I}, \mathcal{J}$, then a *concrete comorphism* from $\mathcal{I}$ to $\mathcal{J}$ is an institution comorphism $(\Phi, \alpha, \beta)\colon \mathcal{I} \longrightarrow \mathcal{J}$ plus a natural transformation $\delta : \; sorts^{\mathcal{I}} \Rightarrow sorts^{\mathcal{J}} \circ \Phi$ and a natural in $\mathcal{I}$-signatures Σ family of natural transformations $\mu_{\Sigma} : \; |\beta_{\Sigma}(\_)|^{\mathcal{I}}_{\Sigma} \Rightarrow (|\_|^{\mathcal{J}}_{\Phi(\Sigma)})\restriction_{\delta_{\Sigma}}$ between functors from $\mathsf{Mod}^{\mathcal{J}}(\Phi(\Sigma))$ to $sorts^{\mathcal{I}}(\Sigma)$-sorted sets, so that for each $\mathcal{I}$-signature Σ, $\Phi(\Sigma)$-model M' in $\mathcal{J}$ and sort $s \in sorts^{\mathcal{I}}(\Sigma)$, we have a function $\mu_{\Sigma,M',s} : \; (|\beta_{\Sigma}(M')|^{\mathcal{I}}_{\Sigma})_s \rightarrow (|M'|^{\mathcal{J}}_{\Phi(\Sigma)})_{\delta_{\Sigma}(s)}$.

Given concrete institutions with symbols $\mathcal{I}$ and $\mathcal{J}$, a *concrete comorphism with symbols* from $\mathcal{I}$ to $\mathcal{J}$ extends an institution comorphism $(\Phi, \alpha, \beta)\colon \mathcal{I} \longrightarrow \mathcal{J}$ by a natural transformation $\delta : \; Symb^{\mathcal{I}} \Rightarrow Symb^{\mathcal{J}} \circ \Phi$ that restricts to $\delta' : \; sorts^{\mathcal{I}} \Rightarrow sorts^{\mathcal{J}} \circ \Phi$, and a family of functions $\mu_{\Sigma} : \; (|\beta_{\Sigma}(\_)|^{\mathcal{I}}_{\Sigma})_s \rightarrow (|\_|^{\mathcal{J}}_{\Phi(\Sigma)})_{\delta'_{\Sigma}(s)}$, required to be natural in $\mathcal{I}$-signatures Σ. Notice that then $(\Phi, \alpha, \beta, \delta', \mu)$ is a concrete comorphism. □

Fact 3.3. An institution comorphism is an isomorphism in $\mathbb{C}oIns$ iff all its components are isomorphisms. □

Unfortunately, institution isomorphism is too strong to capture the notion of "a logic," since it can fail to identify logics that differ only in irrelevant details:

Example 3.4. Let $\mathbf{CPL}'$ be $\mathbf{CPL}$ with arbitrary finite sets as signatures. Then $\mathbf{CPL}'$ has a proper class of signatures, while $\mathbf{CPL}$ only has countably many. Hence, $\mathbf{CPL}$ and $\mathbf{CPL}'$ cannot be isomorphic. □

However, **CPL** and **CPL**$'$ are essentially the same logic. We now give a notion of institution *equivalence* that is weaker than that of institution isomorphism, very much in the spirit of equivalences of categories. The latter weakens isomorphism of categories: two categories are equivalent iff they have isomorphic *skeletons*. A subcategory $\mathbb{S} \hookrightarrow \mathbb{C}$ is a *skeleton* of $\mathbb{C}$ if it is full and each object of $\mathbb{C}$ is isomorphic (in $\mathbb{C}$) to exactly one object in $\mathbb{S}$. In this case, the inclusion $\mathbb{S} \hookrightarrow \mathbb{C}$ has a left inverse (i.e. a retraction) $\mathbb{C} \to \mathbb{S}$ mapping each object to the unique representative of its isomorphism class (see [25]).

Definition 3.5. A (set/cat) institution comorphism (Φ, α, β) is a (set/cat) *institution equivalence* iff

- Φ is an equivalences of categories,
- α_Σ has an inverse up to semantic equivalence α'_Σ, (i.e., $\alpha_\Sigma(\alpha'_\Sigma(\varphi)) \models\mid_\Sigma \varphi$ and $\alpha'_\Sigma(\alpha_\Sigma(\psi)) \models\mid_{\Phi(\Sigma)} \psi$) which is natural in Σ, and
- β_Σ is an equivalence of categories, such that its inverse up to isomorphism and the corresponding isomorphism natural transformations are natural in Σ.

$\mathcal{I}$ is *equivalent* to $\mathcal{J}$ if there is an institution equivalence from $\mathcal{I}$ to $\mathcal{J}$. □

This definition is very natural; it is 2-categorical equivalence in the appropriate 2-category of institutions [10]. The requirement for a set/set institution comorphism to be a *set/set equivalence* is weaker: each β_Σ need only have an inverse up to elementary equivalence β'_Σ.

Definition 3.6. A concrete institution comorphism is a *concrete equivalence* if the underlying institution comorphism is an equivalence and all δ_Σ and $\mu_{\Sigma,M',s}$ are bijective, for each $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$, $M' \in |\mathsf{Mod}^{\mathcal{J}}(\Phi(\Sigma))|$ and $s \in sorts^{\mathcal{I}}(\Sigma)$.

A concrete comorphism with symbols is a *concrete equivalence with symbols* if the underlying institution comorphism is an equivalence and δ_Σ is bijective for each signature Σ. □

Proposition 3.7. Both set/cat and set/set equivalence of institutions are equivalence relations, and set/cat equivalence implies set/set equivalence. □

The following is important for studying invariance properties of institutions under equivalence:

Lemma 3.8. If $(\Phi, \alpha, \beta)\colon \mathcal{I} \longrightarrow \mathcal{J}$ is a set/cat or set/set institution equivalence, $\Gamma \models^{\mathcal{I}}_\Sigma \varphi$ iff $\alpha_\Sigma(\Gamma) \models^{\mathcal{J}}_{\Phi(\Sigma)} \alpha_\Sigma(\varphi)$ for any signature Σ in $\mathcal{I}$ and $\Gamma \cup \{\varphi\} \subseteq \mathsf{Sen}^{\mathcal{I}}(\Sigma)$; also $M_1 \equiv M_2$ iff $\beta_\Sigma(M_1) \equiv \beta_\Sigma(M_2)$, for any $M_1, M_2 \in \mathsf{Mod}^{\mathcal{J}}(\Phi(\Sigma))$. □

Example 3.9. **CPL** and **CPL**$'$ are set/cat equivalent. So are $\mathbf{CPL}^{\neg,\vee,true}$ and $\mathbf{CPL}^{\neg,\wedge,false}$: signatures and models are translated identically, while sentences are translated using de Morgan's laws. Indeed, $\mathbf{CPL}^{\neg,\vee,true}$ and $\mathbf{CPL}^{\neg,\wedge,false}$ are isomorphic, but the isomorphism is far more complicated than the equivalence. □

Definition 3.10. Given a set/cat institution $\mathcal{I}$, an institution $\mathcal{J}$ is a set/cat *skeleton* of $\mathcal{I}$, if

- $\mathbb{S}ign^{\mathcal{J}}$ is a skeleton of $\mathbb{S}ign^{\mathcal{I}}$,
- $\mathsf{Sen}^{\mathcal{J}}(\Sigma) \cong \mathsf{Sen}^{\mathcal{I}}(\Sigma)/{\models\!\mid}$ for $\Sigma \in |\mathbb{S}ign^{\mathcal{J}}|$ (the bijection being natural in Σ), and $\mathsf{Sen}^{\mathcal{J}}(\sigma)$ is the induced mapping between the equivalence classes,
- $\mathsf{Mod}^{\mathcal{J}}(\Sigma)$ is a skeleton of $\mathsf{Mod}^{\mathcal{I}}(\Sigma)$, and $\mathsf{Mod}^{\mathcal{J}}(\sigma)$ is the restriction of $\mathsf{Mod}^{\mathcal{I}}(\sigma)$,
- $M \models^{\mathcal{J}}_{\Sigma} [\varphi]$ iff $M \models^{\mathcal{I}}_{\Sigma} \varphi$.

Set/set skeletons are defined similarly, except that $\mathsf{Mod}^{\mathcal{J}}(\Sigma)$ is in bijective correspondence with $\mathsf{Mod}^{\mathcal{I}}(\Sigma)/\equiv$. □

Theorem 3.11. Assuming the axiom of choice, every institution has a skeleton. Every institution is equivalent to any of its skeletons. Any two skeletons of an institution are isomorphic. Institutions are equivalent iff they have isomorphic skeletons. □

We have now reached a central point, where we can claim

The identity of a logic is the isomorphism type of its skeleton institution.

This isomorphism type even gives a normal form for equivalent logics. It follows that a *property of a logic* must be a property of institutions that is invariant under equivalence, and the following sections explore a number of such properties.

4. Model-Theoretic Invariants of Institutions

This section discusses some model-theoretic invariants of institutions; the table in Thm. 5.20 summarizes the results on this topic in this paper.

Every institution has a Galois connection between its sets Γ of Σ-sentences and its classes $\mathcal{M}$ of Σ-models, defined by $\Gamma^\bullet = \{M \in \mathsf{Mod}(\Sigma) \mid M \models_\Sigma \Gamma\}$ and $\mathcal{M}^\bullet = \{\varphi \in \mathsf{Sen}(\Sigma) \mid \mathcal{M} \models_\Sigma \varphi\}$. A Σ-theory Γ is *closed* if $(\Gamma^\bullet)^\bullet = \Gamma$.[5] Closed Σ-theories are closed under arbitrary intersections; hence they form a complete lattice. This leads to a functor $\mathcal{C}^{\models} \colon \mathbb{S}ign \longrightarrow \mathbb{C}Lat$. Although $\mathcal{C}^{\models}$ is essentially preserved under equivalence, the closure operator $(\_^\bullet)^\bullet$ on theories is not. This means it makes too fine-grained distinctions; for example, in **FOL**, $(true^\bullet)^\bullet$ is infinite, while in a skeleton of **FOL**, $([true]^\bullet)^\bullet$ is just the singleton $\{[true]\}$. As already noted in [33], the closure operator at the same time is too coarse for determining the identity of a logic: while e.g. proof theoretic falsum in a sound and complete logic (see Section 5) is preserved by homeomorphisms of closure operators in the sense of [33], external semantic falsum (see Dfn. 4.2) is not. Because the theory closure operator is not preserved under equivalence, we do not study it further, but instead use the closed theory lattice functor $\mathcal{C}^{\models}$ and the Lindenbaum functor $\mathcal{L}$ defined below. (We note in passing that this Galois connection generalizes some results considered important in the study of ontologies in the computer science sense.)

The category of theories of an institution is often more useful than its lattice of theories, where a theory morphism $(\Sigma, \Gamma) \longrightarrow (\Sigma', \Gamma')$ is a signature morphism $\sigma \colon\ \Sigma \longrightarrow \Sigma$ such that $\Gamma' \models_{\Sigma'} \sigma(\Gamma)$. Let $\mathbb{T}h(\mathcal{I})$ denote this category (it should

[5] The closed theories can serve as models in institutions lacking (non-trivial) models.

be skeletized to become an invariant). The following result is basic for combining theories, and has important applications to both specification and programming languages [19]:

Theorem 4.1. The category of theories of an institution has whatever colimits its category of signatures has. □

Definition 4.2. An institution has *external semantic conjunction* [39] if for any pair of sentences φ_1, φ_2 over the same signature, there is a sentence ψ such that ψ holds in a model iff both φ_1 and φ_2 hold in it. ψ will also be denoted $\varphi_1 \owedge \varphi_2$, a meta-notation which may not agree with the syntax for sentences in the institution. Similarly, one can define what it means for an institution to have external semantic disjunction, negation, implication, equivalence, true, false, and we will use similar circle notations for these. An institution is *truth functionally complete*, if any Boolean combination of sentences is equivalent to a single sentence. □

Example 4.3. **FOL** is truth functionally complete, while **EQ** has no external semantic connectives. □

The Lindenbaum construction of Example 2.2 works for any institution $\mathcal{I}$:

Definition 4.4. Let $\Xi^{\mathcal{I}}$ be the single sorted algebraic signature having that subset of the operations $\{\owedge, \ovee, \ominus, \ominus, \ominus, ⓣ, ⓕ\}$ (with standard arities) that are external semantic for $\mathcal{I}$; $\Xi^{\mathcal{I}}$ may include connectives not provided by the institution $\mathcal{I}$, or provided by $\mathcal{I}$ with a different syntax. We later prove that $\Xi^{\mathcal{I}}$ is invariant under equivalence[6]. For any signature in $\mathcal{I}$, let $\mathcal{L}(\Sigma)$ have as carrier set the quotient $\mathsf{Sen}(\Sigma)/{\models\mid}$, as in Example 2.2. Every external semantic operation of $\mathcal{I}$ has a corresponding operation $\mathcal{L}(\Sigma)$, so $\mathcal{L}(\Sigma)$ can be given a $\Xi^{\mathcal{I}}$-algebra structure. Any subsignature of $\Xi^{\mathcal{I}}$ can also be used (indicated with superscript notation as in Example 2.2), in which case crypto-isomorphisms[7] can provide Lindenbaum algebra equivalence. Moreover, $\mathcal{L}$ is a functor $\mathbb{S}ign \longrightarrow \mathsf{Alg}(\Xi^{\mathcal{I}})$ because $\models\mid$ is preserved by translation along signature morphisms[8]. If $\mathcal{I}$ is truth functionally complete, then $\mathcal{L}(\Sigma)$ is a Boolean algebra. A proof theoretic variant of $\mathcal{L}(\Sigma)$ is considered in Section 5 below. □

Definition 4.5. An institution has *external semantic universal $\mathcal{D}$-quantification* [40] for a class $\mathcal{D}$ of signature morphisms iff for each $\sigma : \Sigma \to \Sigma'$ in $\mathcal{D}$ and each Σ'-sentence, there is a Σ-sentence $\forall\sigma.\varphi$ such that $M \models_\Sigma \forall\sigma.\varphi$ iff $M' \models \varphi$ for each σ-expansion M' of M. External semantic existential quantification is defined similarly. □

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[6] By determining $\Xi^{\mathcal{I}}$ in a purely model-theoretic way, we avoid the need to deal with different signatures of Lindenbaum algebras of equivalent logics, as it is necessary in the framework of [31].

[7] A *cryptomorphism* is a homomorphism between algebras of different signatures linked by a signature morphism; the homomorphism goes from the source algebra into the reduct of the target algebra.

[8] $\mathcal{L}$ is also functorial in the institution, though the details are rather complex.

This definition accommodates quantification over any entities which are part of the relevant concept of signature. For conventional model theory, this includes second order quantification by taking $\mathcal{D}$ to be all extensions of signatures by operation and relation symbols. First order quantification is modeled with $\mathcal{D}$ the *representable signature morphisms* [11, 13] defined below, building on the observation that an assignment for a set of (first order) variables corresponds to a model morphism from the free (term) model over that set of variables:

Definition 4.6. A signature morphism $\chi : \ \Sigma \to \Sigma'$ is *representable* iff there are a Σ-model M_χ called the *representation of* χ and a category isomorphism i_χ such that the diagram below commutes, where $(M_\chi/\mathsf{Mod}(\Sigma))$ is a comma category and U is the forgetful functor. □

$$\begin{array}{ccc} \mathsf{Mod}(\Sigma') & \xrightarrow{i_\chi} & (M_\chi/\mathsf{Mod}(\Sigma)) \\ & \searrow^{\mathsf{Mod}(\chi)} & \downarrow U \\ & & \mathsf{Mod}(\Sigma) \end{array}$$

It seems likely that if external semantic universal quantification over representable quantifiers is included in the signature $\Xi^{\mathbf{FOL}}$, then our Lindenbaum algebra functor yields cylindric algebras, though not all details have been checked as of this writing.

Theorem 4.7. Let $(\Phi, \alpha, \beta) \colon \mathcal{I} \longrightarrow \mathcal{J}$ be an institution equivalence. Then $\mathcal{I}$ has universal (or existential) representable quantification iff $\mathcal{J}$ also has universal (or existential) representable quantification.

It follows that the set of the external semantic connectives an institution has is preserved under institution equivalence.

Example 4.8. Horn clause logic is not equivalent to **FOL**, because it does not have negation (nor implication etc.). Horn clause logic with predicates and without predicates are not equivalent: in the latter logic, model categories of theories have (regular epi, mono)-factorizations, which is not true for the former logic. □

Example 4.9. Propositional logic **CPL** and propositional modal logic **K** are not equivalent: the former has external semantic disjunction, the latter does not: $(M \models_\Sigma p)$ or $(M \models_\Sigma \neg p)$ means that p is interpreted homogeneously in all worlds of M, which is not expressible by a modal formula. Indeed, the Lindenbaum algebra signature for **CPL** is $\{\textcircled{\wedge}, \textcircled{\vee}, \textcircled{\neg}, \textcircled{\Rightarrow}, \textcircled{\Leftrightarrow}, \textcircled{t}, \textcircled{f}\}$, while that for **K** is $\{\textcircled{\wedge}, \textcircled{\Leftrightarrow}, \textcircled{t}, \textcircled{f}\}$. Likewise, first-order logic and first-order modal logic are not equivalent. These assertions also hold replacing "modal" by "intuitionistic". □

Definition 4.10. An institution is liberal iff for any theory morphism $\sigma \colon T_1 \longrightarrow T_2$, $\mathsf{Mod}(\sigma) \colon \mathsf{Mod}(T_2) \longrightarrow \mathsf{Mod}(T_1)$ has a left adjoint. An institution has initial (terminal) models if $\mathsf{Mod}(T)$ has so for each theory T. □

Definition 4.11. For any classes $\mathcal{L}$ and $\mathcal{R}$ of signature morphisms in an institution $\mathcal{I}$, the institution has the *semantic Craig* $\langle \mathcal{L}, \mathcal{R} \rangle$*-interpolation property*[9] [39] if for any pushout

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma_1} & \Sigma_1 \\ {\scriptstyle \sigma_2}\downarrow & & \downarrow{\scriptstyle \theta_2} \\ \Sigma_2 & \xrightarrow{\theta_1} & \Sigma' \end{array}$$

in $\mathbb{S}ign$ such that $\sigma_1 \in \mathcal{L}$ and $\sigma_2 \in \mathcal{R}$, any set of Σ_1-sentences Γ_1 and any set of Σ_2-sentences Γ_2 with $\theta_2(\Gamma_1) \models \theta_1(\Gamma_2)$, there exists a set of Σ-sentences Γ (called the *interpolant*) such that $\Gamma_1 \models \sigma_1(\Gamma)$ and $\sigma_2(\Gamma) \models \Gamma_2$. □

This generalizes the conventional formulation of interpolation from intersection/union squares of signatures to arbitrary classes of pushout squares. While **FOL** has interpolation for all pushout squares [17], many sorted first order logic has it only for those where one component is injective on sorts [8, 6, 23], and **EQ** and Horn clause logic only have it for pushout squares where $\mathcal{R}$ consists of injective morphisms [35, 14], or where L consists of sort-injective morphisms that encapsulate the operation symobls, i.e. no new operation symbol has an old result sort [34]. Using sets of sentences rather than single sentences accommodates interpolation results for equational logic [35] as well as for other institutions having Birkhoff-style axiomatizability properties [14].

Definition 4.12. An institution is *(semi-)exact* if Mod maps finite colimits (pushouts) to limits (pullbacks). □

Semi-exactness is important because many model theoretic results depend on it. It is also important for instantiating parameterized specifications. It means that given a pushout as in Def. 4.11 above, any pair $(M_1, M_2) \in \mathsf{Mod}(\Sigma_1) \times \mathsf{Mod}(\Sigma_2)$ that is *compatible* in the sense that M_1 and M_2 reduce to the same Σ-model can be *amalgamated* to a unique Σ'-model M (i.e., there exists a unique $M \in \mathsf{Mod}(\Sigma')$ that reduces to M_1 and M_2, respectively), and similarly for model morphisms. *Elementary amalgamation* [14] is like semi-exactness but considers the model reducts up to elementary equivalence.

It is also known how to define reduced products, Łoś sentences (i.e. sentences preserved by both ultraproducts and ultrafactors) and Łoś institutions [11], elementary diagrams of models [12], and (Beth) definability[10], all in an institution independent way, such that the expected theorems hold under reasonable assumptions. All this is very much in the spirit of "abstract model theory," in the sense advocated by Jon Barwise [2], but it goes much further, including even some new results for known logics, such as many sorted first order logic [14, 23].

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[9] Recently, it has been noticed that a slightly stronger property, the Craig-Robinson interpolation property, seems to be more appropriate in many contexts; see [15] for details.

[10] Some important results on Beth definability (e.g., the Beth theorem, which asserts the equivalence of explicit and implicit definability) have been stimulated by the first version of the present paper and now have been published, see [32].

The faithful functors to $\mathbb{S}et$ make it possible to consider cardinalities for signatures and models in a concrete institutions with symbols. By restricting signature morphisms to a subcategory, it is often possible to view these cardinality functions as functors.

Definition 4.13. The *Lindenbaum cardinality spectrum* of a concrete institution with symbols maps a cardinal number κ to the maximum number of non-equivalent sentences for a signature of cardinality κ. The *model cardinality spectrum* of a concrete institution with symbols maps each pair of a theory T and a cardinal number κ to the number of non-isomorphic models of T of cardinality κ. □

Theorem 4.14. Sentence and model cardinality spectra, and the finite model property, are preserved under concrete equivalence. □

5. Proof Theoretic Invariants

Proof theoretic institutions include both proofs and sentences. Categorical logic usually works with categories of sentences, where morphisms are (equivalence classes of) proof terms [24]. But this only captures provability between single sentences, whereas logic traditionally studies provability from a *set* of sentences. The following overcomes this limitation by considering categories of sets of sentences:

Definition 5.1. A *cat/cat institution* is like a set/cat institution, except that now $\mathsf{Sen}\colon \mathbb{S}ign \longrightarrow \mathbb{S}et$ comes with an additional categorical structure on sets of sentences, which is carried by a functor $\mathsf{Pr}\colon \mathbb{S}ign \longrightarrow \mathbb{C}at$ such that $(\_)^{op} \circ \mathcal{P} \circ \mathsf{Sen}$ is a subfunctor of Pr, and the inclusion $\mathcal{P}(\mathsf{Sen}(\Sigma))^{op} \hookrightarrow \mathsf{Pr}(\Sigma)$ is broad and preserves products of disjoint sets of sentences[11]. Here $\mathcal{P}\colon \mathbb{S}et \longrightarrow \mathbb{C}at$ is the functor taking each set to its powerset, ordered by inclusion, construed as a thin category[12]. Preservation of products implies that proofs of $\Gamma \to \Psi$ are in bijective correspondence with families of proofs $(\Gamma \to \psi)_{\psi\in\Psi}$, and that there are monotonicity proofs $\Gamma \to \Psi$ whenever $\Psi \subseteq \Gamma$.

A *cat/cat institution comorphism* between cat/cat institutions $\mathcal{I}$ and $\mathcal{J}$ consists of a set/cat institution comorphism $(\Phi, \alpha, \beta)\colon \mathcal{I} \longrightarrow \mathcal{J}$ and a natural transformation $\gamma\colon \mathsf{Pr}^{\mathcal{I}} \longrightarrow \mathsf{Pr}^{\mathcal{J}} \circ \Phi$ such that translation of sentence sets is compatible with translation of single sentences: $\gamma_\Sigma \circ \iota_\Sigma = \iota'_\Sigma \circ \mathcal{P}(\alpha_\Sigma)^{op}$, where ι_Σ and ι'_Σ are the appropriate inclusions. A cat/cat institution comorphism $(\Phi, \alpha, \beta, \gamma)$ is a *cat/cat equivalence* if Φ is an equivalence of categories, β is a family of equivalences natural in Σ, and so is γ. Note that there is no requirement on α. As before, all this also extends to the case of omitting of model morphisms, i.e. the cat/set case. Henceforth, the term *proof theoretic institution* will refer to both the cat/cat and the cat/set cases. □

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[11] Instead of having two functors Pr and Sen, it is also possible to have one functor into a comma category.

[12] A category is *thin* if between two given objects, there is at most one morphism, i.e. the category is a pre-ordered class.

Given an arbitrary but fixed proof theoretic institution, we can define an entailment relation $\vdash_\Sigma$ between sets of Σ-sentences as follows: $\Gamma \vdash_\Sigma \Psi$ if there exists a morphism $\Gamma \to \Psi$ in $\mathsf{Pr}(\Sigma)$. A proof theoretic institution is *sound* if $\Gamma \vdash_\Sigma \Psi$ implies $\Gamma \models_\Sigma \Psi$; it is *complete* if the converse implication holds. In the sequel, we will assume that all proof theoretic institutions are sound, which in particular implies the following:

Proposition 5.2. Any cat/cat equivalence is a set/cat equivalence. □

Proposition 5.3. $\vdash$ satisfies the properties of an *entailment system* [28], i.e. it is reflexive, transitive, monotonic and stable under translation along signature morphisms. In fact, entailment systems are in bijective correspondence with proof theoretic institutions having trivial model theory (i.e. $\mathsf{Mod}(\Sigma) = \emptyset$) and thin categories of proofs. □

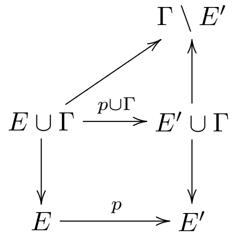
The requirement for sentence translation in proof theoretic institution equivalences is very close to the notion of translational equivalence introduced in [31]. A set/set institution equivalence basically requires that the back-and-forth translation of sentence is semantically equivalent to the original sentence (i.e. $\alpha'_\Sigma(\alpha_\Sigma(\varphi)) \models\!\mid \varphi$); a similar notion would arise when using $\dashv\vdash$. Note, however, that this does not work well for modal logics, since e.g. in **S5**, $\varphi \dashv\vdash \Box\varphi$. Therefore, [31] require $\vdash \alpha'_\Sigma(\alpha_\Sigma(\varphi)) \leftrightarrow \varphi$. However, this is based upon the presence of equivalence as a proof theoretic connective, which is not present in all institutions. Our solution to this problem comes naturally out of the above definition of proof theoretic (i.e. cat/cat or cat/set) equivalence: $\alpha'_\Sigma(\alpha_\Sigma(\varphi))$ and φ have to be isomorphic in the category of proofs. We thus neither identify φ and $\Box\varphi$ in modal logics, nor rely on the presence of a connective $\leftrightarrow$.

Definition 5.4. A proof theoretic institution is *finitary* if $\Gamma \vdash_\Sigma \varphi$ implies $\Gamma' \vdash_\Sigma \varphi$ for some finite $\Gamma' \subseteq \Gamma$.

A proof theoretic institution has *proof theoretic conjunction* if each category $\mathsf{Pr}(\Sigma)$ has distinguished products of singletons, which are singletons again and which are preserved by the proof translations $\mathsf{Pr}(\varphi)$. In terms of derivability, this implies that for φ_1, φ_2 Σ-sentences, there is a product sentence $\varphi_1 \boxed{\wedge} \varphi_2$, and two "projection" proof terms $\pi_1 : \varphi_1 \boxed{\wedge} \varphi_2 \longrightarrow \varphi_1$ and $\pi_2 : \varphi_1 \boxed{\wedge} \varphi_2 \longrightarrow \varphi_2$, such that for any ψ with $\psi \vdash_\Sigma \varphi_1$ and $\psi \vdash_\Sigma \varphi_2$, then $\psi \vdash_\Sigma \varphi_1 \boxed{\wedge} \varphi_2$.

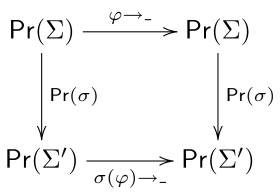
Similarly, a proof theoretic institution has *proof theoretic disjunction (true, false)* if each proof category has distinguished coproducts of singletons that are singletons (a distinguished singleton terminal object, a distinguished singleton initial object) which are preserved by the proof translations.

For each set Γ of Σ-sentences, there is a canonical homomorphism of graphs $\_ \cup \Gamma \colon \mathsf{Pr}(\Sigma) \longrightarrow \mathsf{Pr}(\Sigma)$ as defined by the following commutative diagram of proofs:



(where the monotonicity proofs are not labelled). In general, the graph homomorphism $\_ \cup \Gamma$ is *not* functorial!

A proof theoretic institution has *proof theoretic implication* if each graph homomorphism $\_ \cup \{\varphi\} \colon \mathsf{Pr}(\Sigma) \longrightarrow \mathsf{Pr}(\Sigma)$ has a distinguished "right adjoint", denoted by $\varphi \boxed{\rightarrow} \_$, such that $\varphi \boxed{\rightarrow} \_$ maps singletons to singletons and $\varphi \boxed{\rightarrow} \Gamma = \{\varphi \boxed{\rightarrow} \psi | \psi \in \Gamma\}$ and such that it commutes with the proof translations. This means there exists a bijective correspondence, called the 'Deduction Theorem' in classical logic, between $\mathsf{Pr}(\Sigma)(\Gamma \cup \{\rho\}, E)$ and $\mathsf{Pr}(\Sigma)(\Gamma, \rho \rightarrow E)$ natural in Γ and E, and such that the following diagram commutes for all signature morphisms $\sigma : \Sigma \rightarrow \Sigma'$:



In case $\boxed{f}$ is present, it has *proof theoretic negation* if each sentence ψ has a distinguished negation $\boxed{\neg}\psi$ which is preserved by the proof translations $\mathsf{Pr}(\varphi)$ and such that $\mathsf{Pr}(\Sigma)(\Gamma \cup \{\psi\}, \boxed{f})$ is in natural bijective correspondence to $\mathsf{Pr}(\Sigma)(\Gamma, \{\boxed{\neg}\psi\})$.

A proof theoretic institution is *propositional* if it has proof theoretic conjunction, disjunction, implication, negation, true and false. □

Definition 5.5. A proof theoretic institution with proof theoretic negation has $\neg\neg$-*elimination* if for each Σ-sentence φ, $\boxed{\neg}\,\boxed{\neg}\varphi \vdash_\Sigma \varphi$ (the converse relation easily follows from the definition). □

For example, **CPL** and **FOL** have $\neg\neg$-elimination, while **IPL** has not. Clearly, any complete proof theoretic institution with external semantic and proof theoretic negation has $\neg\neg$-elimination.

Proposition 5.6. A proof theoretic institution having proof theoretic implication enjoys the deduction theorem and modus ponens for $\vdash_\Sigma$. A complete proof theoretic institution is finitary iff it is compact. □

Example 5.7. The modal logic **K** does not have proof theoretic implication, nor negation, and this is a difference from intuitionistic logic **IPL**, showing that the two logics are not equivalent. (See [24] for the proof category of **IPL**.) □

While **K** does not have proof theoretic implication, it still has a form of *local* implication, which does not satisfy the deduction theorem. This can be axiomatized as follows:

Definition 5.8. A proof theoretic institution has *Hilbert implication* if for each signature Σ, there is a unique binary operator $\boxed{\Rightarrow}$ on Σ-sentences satisfying the Hilbert axioms for implication, i.e.

$$\begin{array}{ll} (K) & \emptyset \vdash_\Sigma \{\varphi \boxed{\Rightarrow} \psi \boxed{\Rightarrow} \varphi\} \\ (S) & \emptyset \vdash_\Sigma \{(\varphi \boxed{\Rightarrow} \psi \boxed{\Rightarrow} \chi) \boxed{\Rightarrow} (\varphi \boxed{\Rightarrow} \psi) \boxed{\Rightarrow} \varphi \boxed{\Rightarrow} \chi\} \\ (MP) & \{\varphi \boxed{\Rightarrow} \psi, \varphi\} \vdash_\Sigma \{\psi\} \end{array}$$

□

There is a proof theoretic variant of the Lindenbaum algebra of Def. 4.4:

Definition 5.9. Let $\Psi^{\mathcal{I}}$ be the single sorted algebraic signature having a subset of the operations $\{\boxed{\wedge}, \boxed{\vee}, \boxed{\neg}, \boxed{\Rightarrow}, \boxed{\Leftrightarrow}, \boxed{t}, \boxed{f}\}$ (with their standard arities), chosen according to whether $\mathcal{I}$ has proof theoretic conjunction, disjunction, negation etc., and Hilbert implication for implication. Note that like the signature $\Xi^{\mathcal{I}}$ introduced in Def. 4.4, $\Psi^{\mathcal{I}}$ may include connectives not provided by the institution $\mathcal{I}$, or provided by $\mathcal{I}$ with a different syntax. By Thm. 5.11, $\Xi^{\mathcal{I}}$ has a canonical embedding into $\Psi^{\mathcal{I}}$. Consider $\mathcal{L}^{\vdash\!\dashv}(\Sigma) = \mathsf{Sen}(\Sigma)/\cong$, where $\cong$ is isomorphism in $\mathsf{Pr}(\Sigma)$. Since products etc. are unique up to isomorphism, it is straightforward to make this a $\Psi^{\mathcal{I}}$-algebra.

The Lindenbaum algebra is the basis for the *Lindenbaum category* $\mathcal{LC}^{\vdash\!\dashv}(\Sigma)$, which has object set $\mathcal{P}(\mathcal{L}^{\vdash\!\dashv}(\Sigma))$. By choosing a system of canonical representatives for $\mathsf{Sen}(\Sigma)/\cong$, this object set can be embedded into $|\mathsf{Pr}(\Sigma)|$; hence we obtain an induced full subcategory, which we denote by $\mathcal{LC}^{\vdash\!\dashv}(\Sigma)$. Different choices of canonical representatives may lead to different but isomorphic Lindenbaum categories. While the Lindenbaum category construction is functorial, the proof theoretic Lindenbaum algebra construction is generally not. Also, the closed theory functor $\mathcal{C}^{\models}$ has a proof theoretic counterpart $\mathcal{C}^{\vdash}$ taking theories closed under $\vdash$. □

Definition 5.10. A proof theoretic institution is *compatible* if for each circled (i.e., external semantic) operator in $\Xi^{\mathcal{I}}$, the corresponding boxed (i.e., proof theoretic) operator in $\Psi^{\mathcal{I}}$ is present. It is *bicompatible* if also the converse holds. □

Theorem 5.11. A complete proof theoretic institution with thin proof categories is compatible, but not necessarily bicompatible. □

Proposition 5.12. Assume a proof theoretic institution with thin proof categories. If deduction is complete, then $\mathcal{L}(\Sigma)$ and $\mathcal{L}^{\vdash\!\dashv}(\Sigma)|_{\Xi^{\mathcal{I}}}$ are isomorphic; just soundness gives a surjective cryptomorphism $\mathcal{L}^{\vdash\!\dashv}(\Sigma)|_{\Xi^{\mathcal{I}}} \to \mathcal{L}(\Sigma)$, and just completeness gives one in the opposite direction. □

Example 5.13. Intuitionistic propositional logic shows that proof theoretic disjunction does not imply external semantic disjunction. □

Definition 5.14. A proof theoretic institution is *classical modal* if its Lindenbaum algebras $\mathcal{L}^{\dashv\vdash}(\Sigma)$ are Boolean algebras (also having implication) with an operator $\Box$ (congruent with $\boxed{\wedge}$ and $\boxed{t}$). A classical modal proof theoretic institution is *normal* if the operator satisfies the necessitation law: $\varphi \vdash_\Sigma \Box\varphi$. (Note that modus ponens already follows from implication being present in $\Psi^{\mathcal{I}}$.) □

It is clear that equivalences between classical modal proof theoretic institutions need to preserve $\mathcal{L}^{\dashv\vdash}$ (but not necessarily the operator). It should hence be possibly to apply the results of [31].

Example 5.15. **S4** has a non-idempotent operator (congruent with $\boxed{\wedge}$ and $\boxed{t}$) on its Lindenbaum algebra, while **S5** does not have one. Hence, **S4** and **S5** are not equivalent. □

Definition 5.16. Given cat/cat institutions $\mathcal{I}$ and $\mathcal{J}$, $\mathcal{J}$ is a cat/cat *skeleton* of $\mathcal{I}$ if it is like a set/cat skeleton, but such that $\mathsf{Sen}^{\mathcal{J}}(\Sigma) = \mathsf{Sen}^{\mathcal{I}}(\Sigma)/\cong$, and such that $\mathsf{Pr}^{\mathcal{J}}(\Sigma) = \mathcal{LC}^{\dashv\vdash}(\Sigma)$, the Lindenbaum category. □

Lawvere [26, 27] defined quantification as adjoint to substitution. Here we define quantification as adjoint to sentence translation along a class $\mathcal{D}$ of signature morphisms, which typically introduce new constants to serve as quantification "variables":

Definition 5.17. A cat/cat institution *has proof theoretic universal (existential) $\mathcal{D}$-quantification* for a class $\mathcal{D}$ of signature morphisms stable under pushouts, if for all signature morphisms $\sigma \in \mathcal{D}$, $\mathsf{Pr}(\sigma)$ has a distinguished right (left) adjoint, denoted by $(\forall\sigma)\_$ ($(\exists\sigma)\_$) and preserved by proof translations along signature morphisms. This means there exists a bijective correspondence between $\mathsf{Pr}(\Sigma)(E, (\forall\sigma)E')$ and $\mathsf{Pr}(\Sigma')(\sigma(E), E')$ natural in E and E', in classical logic known as the 'Generalisation Rule', such that for each signature pushout with $\sigma \in \mathcal{D}$,

$$\begin{array}{ccc} \Sigma & \xrightarrow{\theta} & \Sigma_1 \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ \Sigma' & \xrightarrow[\theta']{} & \Sigma'_1 \end{array}$$

the pair $\langle \mathsf{Pr}(\theta), \mathsf{Pr}(\theta') \rangle$ is a morphims of adjunctions. □

One may define a proof theoretic concept of consistency. A theory (Σ, Γ) is *consistent* when its closure under $\vdash$ is a proper subset of $\mathsf{Sen}(\Gamma)$. Craig interpolation also has a proof theoretic version: for any proof $p: \theta_1(E_1) \to \theta_2(E_2)$, there exist proofs $p_1: E_1 \to \sigma_1(E)$ and $p_2: \sigma_2(E) \to E_2$ such that $p = \theta_2(p_2) \circ \theta_1(p_1)$.

Given a set/cat (or set/set) institution $\mathcal{I}$, we can obtain a complete cat/cat (or cat/set) institution $\mathcal{I}^{\models}$ by letting $\mathsf{Pr}(\Sigma)$ be the pre-order defined by $\Gamma \leq \Psi$ if

$\Gamma \models_\Sigma \Psi$, considered as a category. Some of the proof theoretic notions are useful when interpreted in $\mathcal{I}^{\models}$:

Definition 5.18. An institution $\mathcal{I}$ has *internal semantic conjunction* if $\mathcal{I}^{\models}$ has proof theoretic conjunction; similarly for the other connectives. □

Example 5.19. Intuitionistic logic **IPL** has internal, but not external semantic implication. Higher-order intuitionistic logic interpreted in a fixed topos (see [24]) has proof theoretic and Hilbert implication, but neither external nor internal semantic implication. Modal logic **S5** has just Hilbert implication. □

Theorem 5.20. The properties in the table below are invariant under set/set, set/cat, cat/set and cat/cat equivalence, resp.[13] Sect. 5.) Properties in *italics* rely on concrete institutions (as in Def. 2.7).

set/set	set/cat
compactness, (semi-)exactness, elementary amalgamation, semantic Craig interpolation, Beth definability, having external semantic conjunction, disjunction, negation, true, false, being truth functionally complete, Lindenbaum signature $\Xi^{\mathcal{I}}$, Lindenbaum algebra functor $\mathcal{L}$, closed theory lattice functor $\mathcal{C}^{\models}$, (co)completeness of the signature category, *Lindenbaum cardinality spectrum, finite model property.*	all of set/set, having external semantic universal or existential (representable) quantification, exactness, elementary diagrams, (co)completeness of model categories, existence of reduced products, preservation for formulæalong reduced products, being a Łoś-institution, *model cardinality spectrum, admission of free models.*
cat/set	**cat/cat**
all of set/set plus its proof theoretic counterparts where applicable, soundness, completeness, Hilbert implication, $\neg\neg$-elimination.	all of set/cat and cat/set, having proof theoretic universal or existential quantification, compatibility, bicompatibility.

6. $\mathbb{C}/\mathbb{D}$-institutions

In this final section, we take an even more relativistic view on institutions. We already have introduced a number of variants of institutions: set/set, set/cat, cat/set and cat/cat. This can be made more formal in the following way.

[13]Functors such as the Lindenbaum algebra functor are preserved in the sense that $\mathcal{L}_{\mathcal{I}}$ is naturally isomorphic to $\mathcal{L}_{\mathcal{J}} \circ \Phi$.

A category $\mathbb{C}$ is *concrete* [1], if it is equipped with a forgetful functor $|\_|: \mathbb{C} \longrightarrow \mathbb{S}et$.[14]

Given concrete categories $\mathbb{C}$ and $\mathbb{D}$, a $\mathbb{C}/\mathbb{D}$-institution consists of

- a category $\mathbb{S}ign$ of signatures,
- a sentence functor $\mathsf{Sen}: \mathbb{S}ign \to \mathbb{C}$,
- a model functor $\mathsf{Mod}: \mathbb{S}ign^{\mathrm{op}} \to \mathbb{D}$, and
- a satisfaction relation $\models_\Sigma \subseteq |\mathsf{Mod}(\Sigma)| \times |\mathsf{Sen}(\Sigma)|$ for each $\Sigma \in |\mathbb{S}ign|$,

subject to the satisfaction condition: $M' \models_{\Sigma'} |\mathsf{Sen}(\sigma)|(\varphi)$ iff $|\mathsf{Mod}(\sigma)|(M') \models_\Sigma \varphi$ for each $M' \in |\mathsf{Mod}(\Sigma')|$ and $\varphi \in |\mathsf{Sen}(\Sigma)|$.

With this terminology, set/set-institutions are just $\mathbb{S}et/\mathbb{S}et$-institutions, and set/cat-institutions are $\mathbb{S}et/\mathbb{CAT}$-institutions. What about the cat/set and cat/cat. variants?

Let $\mathcal{P}: \mathbb{S}et \longrightarrow \mathbb{S}et$ be the covariant power set functor, and $|\_|: \mathbb{C}at \longrightarrow \mathbb{S}et$ be the functor sending each small category to its set of objects. Then $\mathbb{P}owerCat$, the comma category $(\mathcal{P} \downarrow |\_|)$, consists of small categories whose set of objects is the power set of a given set (of "sentences"), and functors whose object mapping is induced by a mapping between the sets of sentences. Our cat/cat-institutions are more precisely $\mathbb{P}owerCat/\mathbb{CAT}$-institutions; a similar remark holds for cat/set-institititions.

Florian Rabe (personal communication) pointed out the following phenomenon: if a proof $p: \{\varphi, \psi\} \longrightarrow \{\chi\}$ is translated along a signature morphism σ that identifies φ and ψ, we get a proof $\sigma(p): \{\sigma(\varphi)\} \longrightarrow \{\sigma(\chi)\}$ where the information about the original numbers of premises is lost. In some cases (for example, when generating free proof systems [16, 15]), it is desirable to have the facility to keep this multiplicity information[15], which amounts to work with *multisets* of sentences (instead of just sets) in the premises of proofs. A more algebraic way of formulating this is the following

Proof categories for Lawvere/cat-institutions are many-sorted Lawvere theories

offering a new variant of the Curry-Howard isomorphism.

Recall that a single-sorted finitary Lawvere theory [26, 27] is a product-preserving object-bijective functor from $\mathbb{N}^{op}$ to $\mathbb{P}$, where $\mathbb{N}$ is the category of finite ordinals (considered as a subcategory of $\mathbb{S}et$) and $\mathbb{P}$ is any category.(of "term tuples"). Now a many.sorted Lawvere theory (*without rank*) is a product-preserving object-bijective functor from $\mathbb{F}am(S)^{op}$ to $\mathbb{P}$. Here, for a set S, $\mathbb{F}am(S)$ is the comma category $(\mathbb{S}et, S)$ of families $\phi: X \longrightarrow S$ of elements of S (where X is an arbitrary index set) and reindexing morphisms. $\mathbb{F}am(S)$ can be shown to be equivalent to the free category with coproducts over the set S of generating objects.

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[14] The use of the same symbol $|\_|$ for the forgetful functor here and for the collection of objects of a category should not be confusing. Moreover, in some cases (for example, in the case of $\mathbb{CAT}$), we need to replace the category of sets by the quasi-category of classes.

[15] The problem is that otherwise disjointness of sentence sets are not preserved, and hence products (=disjoint unions) are not preserved either

Notice that a family $\phi\colon X \longrightarrow S$ a family is nothing but an S-sorted system of variables; a proof judgement hence is a variable context. Each variable can be thought of as standing for an unknown proof, and the type of the variable being the sentence being witnessed by the proof. This nicely corresponds to the intuition (guided by many examples, and also by the standard interpretation of algebraic theories as Lawvere theories) that the category of proofs has as morphisms tuples of terms with variables, where the variable context is given by the source proof judgement. Further note that for a sentence ψ, the projections $\pi_1, pi_2\colon \psi \times \psi \longrightarrow \psi$ are retractions with one-sided inverse $\langle id, id \rangle$, but they are in general not isomorphisms. This exactly yields the difference between ψ and $\psi \times \psi$ needed for the multiset-character of proof judgements.

Though it has to be elaborated whether the concept of Lawvere/cat institution works well with the examples, the naturalness and elegance of the concepts already sounds very promising.

7. Conclusions

We believe this paper has established four main points: (1) The notion of "a logic" should depend on the purpose at; in particular, proof theory and model theory sometimes treat essentially the same issue in different ways. Institutions provide an appropriate framework, having a balance between model theory and proof theory. (2) Every plausible notion of equivalence of logics can be formalized using institutions and various equivalence relations on them. (3) Inequivalence of logics can be established using various constructions on institutions that are invariant under the appropriate equivalence, such as Lindenbaum algebras and cardinality spectra. We have given several examples of such inequivalences. (4) A great deal of classical logic can be generalized to arbitrary institutions, and the generalized formulations are often quite interesting in themselves. Perhaps the fourth point is the most exciting, as there remains a great deal more to be done, particularly in the area of proof theory.

Among the proof theoretic properties that we have not treated. Proof theoretic ordinals, while an important device, would deviate a bit from the subject of this paper, because they are a measure for the proof theoretic strength of a *theory* in a logic, not a measure for the logic itself. But properties like (strong) normal forms for proofs could be argued to contribute to the identity of a logic; treating them would require $\mathsf{Pr}(\Sigma)$ to become an order-enriched category or a 2-category of sentences, with proof terms and proof term reductions. A related topic is cut elimination, which would require an even finer structure on $\mathsf{Pr}(\Sigma)$, with proof rules of particular format. Another direction is the introduction of numberings in order to study recursiveness of entailment. We hope this paper provides a good starting point for such investigations.

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