Proof Systems for Institutional Logic

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Abstract

Institutions with proof-theoretic structure, here called 'institutions with proofs', provide a complete formal notion for the intuitive notion of logic, including both the model and the proof theoretic sides. This paper introduces a concept of proof rules for institutions and argues that the proof systems of the actual institutions with proofs are freely generated by their presentations as systems of proof rules. We also show that proof-theoretic quantification, an institutional refinement of the (meta-)rule of Generalization from classical logic, can also be added freely to any proof system. By applying these universal properties, we are able to provide some general compactness results for proof systems and some general soundness results for institutions with proofs. We also discuss several open problems and further research directions.

Keywords: Institutions, proofs systems.

1 Introduction

The theory of institutions [14] is a categorical abstract model theory which formalizes the intuitive notion of logical system, including syntax, semantics, and the satisfaction relation between them. It provides the most complete form of abstract model theory, the only one including signature morphisms, model reducts, and even morphisms between logics as primary concepts. The concept of institution arose in computing science (algebraic specification) in response to the population explosion among logics in use there, with the ambition of doing as much as possible at a level of abstraction independent of commitment to any particular logic [14, 27, 11]. Besides its extensive use in specification theory (it has become the most fundamental mathematical structure in algebraic specification theory), there have been several substantial developments towards an 'institution-independent' model theory [29, 30, 8, 10, 9, 17, 25].

To handle proof theory, [23] adds proof-theoretic structure to institutions by using an extension of traditional categorical logic [18] with sets of sentences as objects instead of single sentences, and with morphisms representing (equivalence classes of) proofs as usual. This proof-theoretic structure, here called 'proof system', is significantly more refined than the entailment systems of [21] or the π-institutions of [13] by discriminating between different proofs $\Gamma \to E$. However our work avoids a simplistic amalgamation of (institutional) model theoretic and proof theoretic traditions, often based on rather conflicting views of logic phenomena. Therefore we chose to give a fresh approach to proof theory which fits the model theoretic institutional culture.

The proof-theoretic structure of actual logics is almost always presented in the form of systems of (finitary) proof rules rather than proof systems. Our paper clarifies this issue by introducing a formal concept of systems of proof rules for institutions and by showing that systems of proof rules generate freely proof systems. This adjunction result is based on an algebra of proofs. We argue that actual proof systems for institutions are freely generated by corresponding systems of proof rules.

The next part of the paper deals with proof-theoretic quantification as defined by [23] by refining the classical meta-rule of 'Generalization'

$$E \vdash (\forall\chi)\rho \text{ if and only if } E \vdash \rho$$

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doi:10.1093/logcom/exi082

to a property of the proof system quantification rather than considering it as a proof rule. Here we show that (universal) quantification can be added freely to any proof system such that its sentence part has a syntax for quantifiers, and argue that this is the way one obtains the actual proof systems with 'Generalization' as a meta-rule.

We show how these two universal properties lead to

- the compactness of the proof system freely generated by a system of finitary proof rules,
- the automatic transfer of compactness from proof systems to proof systems with universal quantification,
- the automatic transfer of soundness from institutions with proof rules to institutions with proofs, and
- under the assumption of semantic quantification, further transfer of soundness to institutions with proof having proof-theoretic quantifiers.

In the final section we discuss some open problems regarding proof-theoretic aspects of institutions.

2 Institutions

We assume the reader is familiar with basic notions and standard notations from category theory; see [20] for an introduction to this subject. By way of notation, $|\mathbb{C}|$ denotes the class of objects of a category $\mathbb{C}$, $\mathbb{C}(A,B)$ the set of arrows with domain A and codomain B, and composition is denoted by ';' and in diagrammatic order. The category of sets (as objects) and functions (as arrows) is denoted by $\mathbb{S}et$, and $\mathbb{CAT}$ is the category of all categories.[1] The opposite of a category $\mathbb{C}$ (obtained by reversing the arrows of $\mathbb{C}$) is denoted $\mathbb{C}^{\rm op}$.

An *institution* $\mathcal{I} = (\mathbb{S}ign^{\mathcal{I}}, \mathsf{Sen}^{\mathcal{I}}, \mathsf{Mod}^{\mathcal{I}}, \models^{\mathcal{I}})$ consists of

1. a category $\mathbb{S}ign^{\mathcal{I}}$, whose objects are called *signatures*,
2. a functor $\mathsf{Sen}^{\mathcal{I}}: \mathbb{S}ign^{\mathcal{I}} \to \mathbb{S}et$, giving for each signature a set whose elements are called *sentences* over that signature,
3. a functor $\mathsf{Mod}^{\mathcal{I}}: (\mathbb{S}ign^{\mathcal{I}})^{\rm op} \to \mathbb{CAT}$ giving for each signature Σ a category whose objects are called Σ-*models*, and whose arrows are called Σ-(*model*) *morphisms*, and
4. a relation $\models^{\mathcal{I}}_{\Sigma} \subseteq |\mathsf{Mod}^{\mathcal{I}}(\Sigma)| \times \mathsf{Sen}^{\mathcal{I}}(\Sigma)$ for each $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$, called Σ-*satisfaction*,

such that for each morphism $\varphi: \Sigma \to \Sigma'$ in $\mathbb{S}ign^{\mathcal{I}}$, the *satisfaction condition*

$$M' \models^{\mathcal{I}}_{\Sigma'} \mathsf{Sen}^{\mathcal{I}}(\varphi)(\rho) \text{ iff } \mathsf{Mod}^{\mathcal{I}}(\varphi)(M') \models^{\mathcal{I}}_{\Sigma} \rho$$

holds for each $M' \in |\mathsf{Mod}^{\mathcal{I}}(\Sigma')|$ and $\rho \in \mathsf{Sen}^{\mathcal{I}}(\Sigma)$. When there is no danger of ambiguity, we may skip the superscripts from the notation of the entities of the institution, for example $\mathbb{S}ign^{\mathcal{I}}$ may be simply denoted as $\mathbb{S}ign$.

We denote the *reduct* functor $\mathsf{Mod}^{\mathcal{I}}(\sigma)$ by $\_\upharpoonright_{\varphi}$ and the sentence translation $\mathsf{Sen}^{\mathcal{I}}(\varphi)$ by $\varphi(\_)$. When $M = M'\upharpoonright_{\varphi}$ we say that M is a φ-*reduct of* M', and that M' is a φ-*expansion of* M.

For any signature Σ in an institution $\mathcal{I}$, a Σ-*theory* is any set of Σ-sentences.

- For each Σ-theory E, let $E^* = \{M \in \mathsf{Mod}(\Sigma) \mid M \models_{\Sigma} e \text{ for each } e \in E\}$.
- For each class $\mathcal{M}$ of Σ-models, let $\mathcal{M}^* = \{e \in \mathsf{Sen}(\Sigma) \mid M \models_{\Sigma} e \text{ for each } M \in \mathcal{M}\}$.

[1] Strictly speaking, this is only a quasi-category living in a higher set-theoretic universe.

If E and E' are theories of the same signature, then $E' \subseteq E^{**}$ is denoted by $E \models E'$.

A *theory morphism* $\varphi : (\Sigma, E) \to (\Sigma', E')$ is just a signature morphism $\varphi : \Sigma \to \Sigma'$ such that $E' \models \varphi(E)$.

EXAMPLE 2.1

Let $\mathbf{FOL}$ be the institution of *many sorted first-order logic with equality*. Its signatures (S, F, P) consist of a set of sort symbols S, a set F of function symbols, and a set P of relation symbols. Each function or relation symbol comes with a string of argument sorts, called *arity*, and for functions symbols, a result sort. $F_{w \to s}$ denotes the set of function symbols with arity w and sort s, and P_w the set of relation symbols with arity w.

Signature morphisms map the three components in a compatible way. Models M are first-order structures interpreting each sort symbol s as a set M_s, each function symbol σ as a function M_σ from the product of the interpretations of the argument sorts to the interpretation of the result sort, and each relation symbol π as a subset M_π of the product of the interpretations of the argument sorts. Sentences are the usual first-order sentences built from equational and relational atoms by iterative application of logical connectives and quantifiers. Sentence translations rename the sorts, function, and relation symbols. For each signature morphism φ, the reduct $M'\restriction_\varphi$ of a model M' is defined by $(M'\restriction_\varphi)_x = M'_{\varphi(x)}$ for each x sort, function, or relation symbol from the domain signature of φ. The satisfaction of sentences by models is the usual Tarskian satisfaction defined inductively on the structure of the sentences.

The institution $\mathbf{PL}$ of *propositional logic* is obtained as the sub-institution of $\mathbf{FOL}$ obtained by considering only the empty sorted signatures.

A *universal Horn sentence* in $\mathbf{FOL}$ for a first-order signature (S, F, P) is a sentence of the form $(\forall X)H \Rightarrow C$, where H is a finite conjunction of (relational or equational) atoms and C is a (relational or equational) atom, and $H \Rightarrow C$ is the implication of C by H. The sub-institution $\mathbf{HCL}$, *Horn clause logic*, of $\mathbf{FOL}$ has the same signatures and models as $\mathbf{FOL}$ but only universal Horn sentences as sentences.

An *algebraic signature* (S, F) is just a $\mathbf{FOL}$ signature without relation symbols. The sub-institution of $\mathbf{HCL}$ which restricts the signatures only to the algebraic ones and the sentences to universally quantified equations is called *equational logic* and is denoted by $\mathbf{EQL}$.

EXAMPLE 2.2

The institution $\mathbf{PA}$ of partial algebra [5] (which will be used as a meta-logic in some of the proofs of our paper) is defined as follows.

A *partial algebraic signature* is a tuple (S, TF, PF), where TF is the set of *total* operations and PF is the set of *partial* operations.

A partial algebra is just like an ordinary algebra but interpreting the operations of PF as partial rather than total functions. A *partial algebra homomorphism* $h : A \to B$ is a family of (total) functions $\{h_s : A_s \to B_s\}_{s \in S}$ indexed by the set of sorts S of the signature such that $h_w(A_\sigma(a)) = B_\sigma(h_s(a))$ for each operation $\sigma : w \to s$ and each string of arguments $a \in A_w$ for which $A_\sigma(a)$ is defined.

The sentences have three kinds of atoms: *definedness* def, *strong* equality $\stackrel{s}{=}$, and *existence* equality $\stackrel{e}{=}$. The definedness $def(t)$ of a term t holds in a partial algebra A when the interpretation A_t of t is defined. The strong equality $t \stackrel{s}{=} t'$ holds when both terms are undefined or both of them are defined and are equal. The existence equality $t \stackrel{e}{=} t'$ holds when both terms are defined and are equal. The sentences are formed from these atoms by logical connectives and quantification over total variables.

The Horn sentences in $\mathbf{PA}$ formed from existence equality atoms are called *quasi-existence equations*. Recall [5] that any set of quasi-existence equations admits an initial partial algebra; more generally, each partial algebra admits a free extension along any morphism of quasi-existence equational theories [26, 7].

Other examples of institutions in use in computing science include rewriting [22], higher-order [4], polymorphic [28], temporal [12], process [12], behavioural [3], coalgebraic [6] and object-oriented [15] logics.

In the literature there are several concepts of structure preserving mappings between institutions, which are used for relating different institutions for different purposes such as comparison between institutions, encoding of one institution into another, putting institutions together in a heterogenuous multi-logic environment, borrowing properties from one institution to another, etc.

The original institution mapping, called 'institution morphism' introduced by [14], is adequate for expressing a 'forgetful' operation from a 'more complex' institution to a 'simpler' one. Although the work of our paper can be presented by using institution morphisms, we think that 'institution comorphisms' [16], previously know as 'plain maps' in [21] or 'representations' in [31, 32], give a better intuition with respect to the adjunctions developed in our paper. They capture the idea of embedding a a 'simpler' institution into a 'more complex' one.

An *institution comorphism* $(\Phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$ consists of

1. a functor $\Phi : \mathbb{S}ign \to \mathbb{S}ign'$,
2. a natural transformation $\alpha : \mathsf{Sen} \Rightarrow \Phi; \mathsf{Sen}'$, and
3. a natural transformation $\beta : \Phi^{\mathrm{op}}; \mathsf{Mod}' \Rightarrow \mathsf{Mod}$

such that the following *satisfaction condition* holds

$$M' \models'_{\Phi(\Sigma)} \alpha_\Sigma(e) \;\text{ iff }\; \beta_\Sigma(M') \models_\Sigma e$$

for each signature $\Sigma \in |\mathbb{S}ign|$, for each $\Phi(\Sigma)$-model M', and each Σ-sentence e.

EXAMPLE 2.3
The canonical embedding of equational logic $\mathbf{EQL}$ into first-order logic can be expressed as a comorphism $(\Phi, \alpha, \beta) : \mathbf{EQL} \to \mathbf{FOL}$ such that $\Phi(S, F) = (S, F, \emptyset)$, α regards any equation as a first order sentence, and $\beta_{(S,F)} : \mathsf{Mod}^{\mathbf{FOL}}(S, F, \emptyset) \to \mathsf{Mod}^{\mathbf{EQL}}(S, F)$ is the trivial isomorphism which regards any $(S, F, \emptyset)$-model as an (S, F)-algebra.

EXAMPLE 2.4
$\mathbf{EQL}$ can embedded into the institution $\mathbf{PA}$ of partial algebras by means of the canonical comorphism which maps an algebraic signature (S, F) to the partial algebra signature $(S, F, \emptyset)$.

As mentioned above, everything in this paper can be done with institution morphism instead of comorphisms. The reason for this lies in the one–one canonical correspondence between institution morphisms $(\Phi, \alpha, \beta) : \mathcal{I}' \to \mathcal{I}$ and comorphisms $(\overline{\Phi}, \overline{\alpha}, \overline{\beta}) : \mathcal{I} \to \mathcal{I}'$ when Φ and $\overline{\Phi}$ form a pair of adjoint functors (Φ right, $\overline{\Phi}$ left) (see [2]). As will be seen below in the paper, the relevant corresponding signature functors are in fact identities.

3 Proofs and proof rules

Categorical logic usually works with categories of sentences, where morphisms are (equivalence classes of) proof terms [18]. But this only captures proofs between single sentences, whereas logic traditionally studies proofs from a *set* of sentences. The following definition overcomes this limitation by considering categories of sets of sentences. The multi-signature aspect of institutions is also present, with proof translations corresponding to signature morphisms and which are coherent with respect to the corresponding sentence translations.

DEFINITION 3.1
A *proof system* $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf})$ consists of

- a category of 'signatures' $\mathbb{S}ign$,
- a 'sentence functor' $\mathsf{Sen} : \ \mathbb{S}ign \to \mathbb{S}et$, and
- a 'proof functor' $\mathsf{Pf} : \ \mathbb{S}ign \to \mathbb{CAT}$ (giving for each signature Σ the category of the Σ-proofs)

such that $\mathsf{Sen}; \mathcal{P}; (\_)^{\mathrm{op}}$ is a sub-functor of Pf, and the inclusion $\mathcal{P}(\mathsf{Sen}(\Sigma))^{\mathrm{op}} \hookrightarrow \mathsf{Pf}(\Sigma)$ is broad and preserves finite products of disjoint sets (of sentences) for each signature Σ, where $\mathcal{P} : \ \mathbb{S}et \to \mathbb{CAT}$ is the ($\mathbb{CAT}$-valued) power-set functor.

$$\begin{tikzcd}
\mathsf{Pf}(\Sigma) \arrow[r, "\mathsf{Pf}(\varphi)"] & \mathsf{Pf}(\Sigma') \\
\mathcal{P}(\mathsf{Sen}(\Sigma))^{\mathrm{op}} \arrow[u, hook] \arrow[r, "\mathcal{P}(\mathsf{Sen}(\varphi))^{\mathrm{op}}"'] & \mathcal{P}(\mathsf{Sen}(\Sigma'))^{\mathrm{op}} \arrow[u, hook]
\end{tikzcd}$$

REMARK 3.2
$\mathsf{Pf}(\Sigma)$ has the subsets of $\mathsf{Sen}(\Sigma)$ as objects. Preservation of products implies that there are distinguished monotonicity proofs $\supseteq_{\Gamma,E} : \ \Gamma \to E$ whenever $E \subseteq \Gamma$ which are preserved by signature morphisms, i.e. $\varphi(\supseteq_{\Gamma,E}) = \supseteq_{\varphi(\Gamma),\varphi(E)}$, and that proofs $\Gamma \to E_1 \uplus E_2$ are in one–one natural correspondence with pairs of proofs $\langle \Gamma \to E_1, \, \Gamma \to E_2 \rangle$.[2]

DEFINITION 3.3
A set of sentences Γ *entails* another set of sentences E, denoted $\Gamma \vdash E$, when there exists at least one proof $\Gamma \to E$. Thin proof systems, i.e. such that $\mathsf{Pf}(\Sigma)$ are preorders, are called *entailment systems*.

Note that entailment systems of Definition 3.3 are only slightly more general than those of [21] in the sense that our entailment systems admit multi-conclusion entailments. On the other hand, our proof systems (Definition 3.1) constitute a very different approach to institutional proof theory than the 'proof calculi' of [21].

REMARK 3.4
Any institution $\mathcal{I}$ determines a 'semantic' entailment system, where for each signature $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$, $\mathsf{Pf}(\Sigma)$ is the preorder given by $\Gamma \to E$ if and only if $\Gamma^* \subseteq E^*$. This shows that proof systems are more abstract than institutions.

Conversely, by following a construction from [9], each proof system determines an 'entailment institution' whose models are (proof theoretic) theories. This is an adjoint construction to the forgetful functor from institutions with elementary diagrams to entailment systems.

DEFINITION 3.5
A *proof system comorphism* between proof systems $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf})$ and $(\mathbb{S}ign', \mathsf{Sen}', \mathsf{Pf}')$ consists of

- a 'signature' functor $\Phi : \ \mathbb{S}ign \to \mathbb{S}ign'$,
- a 'sentence translation' natural transformation $\alpha : \ \mathsf{Sen} \Rightarrow \Phi; \mathsf{Sen}'$, and

[2] $E_1 \uplus E_2$ denotes the union of *disjoint* sets E_1 and E_2.

- a 'proof translation' natural transformation $\gamma\colon \mathsf{Pf} \longrightarrow \Phi;\mathsf{Pf}'$ such that translation of sentence sets is compatible with translation of single sentences:

$$\begin{tikzcd}
\mathsf{Pf}(\Sigma) \arrow[r, "\gamma_\Sigma"] & \mathsf{Pf}'(\Phi(\Sigma)) \\
\mathsf{Sen}(\Sigma) \arrow[u, hook] \arrow[r, "\alpha_\Sigma"'] & \mathsf{Sen}'(\Phi(\Sigma)) \arrow[u, hook]
\end{tikzcd}$$

With the obvious composition, proof systems comorphisms form a category denoted $co\mathbb{P}f\mathbb{S}ys$.

The actual proof systems are presented as systems of proof rules; we argue that they are *freely generated* by such presentations.

DEFINITION 3.6
A *system of proof rules* $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c)$ consists of

- a category of 'signatures' $\mathbb{S}ign$,
- a 'sentence functor' $\mathsf{Sen}:\ \mathbb{S}ign \to \mathbb{S}et$,
- a 'proof rule functor' $\mathsf{Rl}:\ \mathbb{S}ign \to \mathbb{S}et$, and
- two natural transformations $h, c:\ \mathsf{Rl} \Rightarrow \mathsf{Sen};\mathcal{P}$, where $\mathcal{P}:\ \mathbb{S}et \to \mathbb{S}et$ is the $\mathbb{S}et$-valued power-set functor.

REMARK 3.7
In the actual systems of proof rules, for each signature Σ, $\mathsf{Rl}(\Sigma)$ gives the set of the Σ-proof rules, $h_\Sigma:\ \mathsf{Rl}(\Sigma) \to \mathcal{P}(\mathsf{Sen}(\Sigma))$ gives the hypotheses of the rules, and $c_\Sigma:\ \mathsf{Rl}(\Sigma) \to \mathcal{P}(\mathsf{Sen}(\Sigma))$ gives the conclusions. A Σ-rule r can therefore be written as $h_\Sigma(r) \xrightarrow{r} c_\Sigma(r)$. The functoriality of Rl and the naturality of the hypotheses h and of the conclusions c, say that the translation of rules along signature morphisms is coherent to the translation of the sentences.

Note that proof rules of Definition 3.6 admit multiple conclusions, which constitute a slight generalization of the usual practice in actual logics which uses only single conclusion rules.

Sometimes, in actual situations, systems of proof rules are defined as signature indexed families $\{rl(\Sigma)\}_{\Sigma\in|\mathbb{S}ign|}$ with $rl(\Sigma) \subseteq \mathcal{P}(\mathsf{Sen}(\Sigma)) \times \mathcal{P}(\mathsf{Sen}(\Sigma))$. Notice that this can be extended canonically to a proper system of proof rules.

Below there are examples of systems of proof rules for two very classical logics.

EXAMPLE 3.8
A system of proof rules for propositional logic $\mathbf{PL}$ is shown in the following table:

$h_\Sigma(r)$	$c_\Sigma(r)$
$\emptyset$	$\{\pi_1 \Rightarrow (\pi_2 \Rightarrow \pi_1)$, $(\pi_1 \Rightarrow (\pi_2 \Rightarrow \pi_3)) \Rightarrow ((\pi_1 \Rightarrow \pi_2) \Rightarrow (\pi_1 \Rightarrow \pi_3))$, $(\neg\pi_1 \Rightarrow \neg\pi_2) \Rightarrow (\pi_2 \Rightarrow \pi_1)$, $(\pi_2 \Rightarrow \pi_1) \Rightarrow (\neg\pi_1 \Rightarrow \neg\pi_2)\}$
$\{\pi_1, \pi_1 \Rightarrow \pi_2\}$	$\{\pi_2\}$

for all $\mathbf{PL}$-signatures Σ and all $\pi_1, \pi_2, \pi_3 \in \Sigma$.

EXAMPLE 3.9
The classical proof rules for equational logic $\mathbf{EQL}$ are as follows:

$h_{(S,F)}(r)$	$c_{(S,F)}(r)$
$\emptyset$	$\{(\forall X)t = t\}$
$\{(\forall X)t = t'\}$	$\{(\forall X)t' = t\}$
$\{(\forall X)t = t', (\forall X)t' = t''\}$	$\{(\forall X)t = t''\}$
$\{(\forall X)t = t'\}$	$\{(\forall Y)\theta(t) = \theta(t') \mid \theta$ substitution $X \to T_F(Y)\}$
$\{(\forall X)t_i = t'_i \mid 1 \leq i \leq n\}$	$\{(\forall X)\sigma(t_1, \ldots, t_n) = \sigma(t'_1, \ldots, t'_n) \mid \sigma \in F_{s_1 \ldots s_n \to s}\}$

for any **EQL** signature (S, F), for all sets of variables X, Y, for all sorts $s_1, \ldots, s_n, s$, and for all F-terms with variables X such that the sort of t_i is s_i for each $1 \leq i \leq n$.

DEFINITION 3.10
A *comorphism of systems of proof rules* between systems of proof rules $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c)$ and $(\mathbb{S}ign', \mathsf{Sen}', \mathsf{Rl}', h', c')$ consists of

- a 'signature' functor $\Phi : \mathbb{S}ign \to \mathbb{S}ign'$,
- a 'sentence translation' natural transformation $\alpha : \mathsf{Sen} \Rightarrow \Phi; \mathsf{Sen}'$,
- a 'rule translation' natural transformation $\gamma : \mathsf{Rl} \Rightarrow \Phi; \mathsf{Rl}'$ which is compatible with the hypotheses and the conclusions, i.e. the diagram below commutes:

$$\begin{tikzcd}
\mathsf{Rl} \arrow[r, Rightarrow, "\gamma"] \arrow[d, Rightarrow, shift right, "h"'] \arrow[d, Rightarrow, shift left, "c"] & \Phi; \mathsf{Rl}' \arrow[d, Rightarrow, shift right, "h'"'] \arrow[d, Rightarrow, shift left, "c'"] \\
\mathsf{Sen}; \mathcal{P} \arrow[r, Rightarrow, "\alpha"'] & \Phi; \mathsf{Sen}'; \mathcal{P}
\end{tikzcd}$$

With the obvious composition, proof systems comorphisms form a category denoted $co\mathbb{R}l\mathbb{S}ys$.

FACT 3.11
We can easily notice that each proof system $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf})$ can be seen as a system of proof rules $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}, dom, cod)$ by regarding each proof as a proof rule (the hypotheses being given by the domain of the proof, and the conclusions by the codomain). This yields a forgetful functor from the category of proof systems $co\mathbb{P}f\mathbb{S}ys$ to the category of proof rule systems $co\mathbb{R}l\mathbb{S}ys$.

The actual proof systems are usually presented by systems of proof rules which generate them freely.

THEOREM 3.12
Each system of proof rules such that its sentence translations are injective generates freely a proof system.

PROOF. Let $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c)$ be a system of proof rules such that $\mathsf{Sen}(\varphi)$ is injective for each signature morphism $\varphi \in \mathbb{S}ign$. We fix a signature $\Sigma \in |\mathbb{S}ign|$ and define the one-sorted **PA** signature consisting of the following:

- total constants, all sets of sentences $E \subseteq \mathsf{Sen}(\Sigma)$, all sets of sentences inclusions $E \supseteq E'$, and all elements of $\mathsf{Rl}(\Sigma)$,
- unary total operation symbols, h and c, and
- binary partial operation symbols, $\_;\_$ and $\langle\_,\_\rangle$, and

Let $E \xrightarrow{p} \Gamma$ abbreviate $(h(p) \stackrel{e}{=} E) \wedge (c(p) \stackrel{e}{=} \Gamma)$.

We consider the initial partial algebra PT^{Σ} of the following set of quasi-existence equations:

(R^{Σ}) $\quad h_{\Sigma}(r) \xrightarrow{r} c_{\Sigma}(r)$
for all $r \in \mathsf{Rl}(\Sigma)$

(S^{Σ}) $\quad E \xrightarrow{E} E$
for all $E \subseteq \mathsf{Sen}(\Sigma)$

$(M1^{\Sigma})$ $\quad (E \supseteq E) \stackrel{e}{=} E$
for all $E \subseteq \mathsf{Sen}(\Sigma)$

$(M2^{\Sigma})$ $\quad E \xrightarrow{E \supseteq E'} E'$
for all $E' \subseteq E \subseteq \mathsf{Sen}(\Sigma)$

$(M3^{\Sigma})$ $\quad (E \supseteq E'); (E' \supseteq E'') \stackrel{e}{=} (E \supseteq E'')$
for all $E'' \subseteq E' \subseteq E \subseteq \mathsf{Sen}(\Sigma)$

$(C1^{\Sigma})$ $\quad (\forall p, p')(E \xrightarrow{p} E') \wedge (E' \xrightarrow{p'} E'') \Rightarrow (E \xrightarrow{p;p'} E'')$
for all $E, E', E'' \subseteq \mathsf{Sen}(\Sigma)$

$(C2^{\Sigma})$ $\quad (\forall p, p', p'')(E \xrightarrow{p} E') \wedge (E' \xrightarrow{p'} E'') \wedge (E'' \xrightarrow{p''} E''') \Rightarrow p; (p'; p'') \stackrel{e}{=} (p; p'); p''$
for all $E, E', E'', E''' \subseteq \mathsf{Sen}(\Sigma)$

$(C3^{\Sigma})$ $\quad (\forall p) E \xrightarrow{p} E' \Rightarrow (E; p \stackrel{e}{=} p) \wedge (p; E' \stackrel{e}{=} p)$
for all $E, E' \subseteq \mathsf{Sen}(\Sigma)$

$(P1^{\Sigma})$ $\quad (\forall p, p')(E \xrightarrow{p} \Gamma) \wedge (E \xrightarrow{p'} \Gamma') \Rightarrow$
$\Rightarrow (E \xrightarrow{\langle p, p' \rangle} \Gamma \cup \Gamma') \wedge (\langle p, p' \rangle; (\Gamma \cup \Gamma' \supseteq \Gamma) \stackrel{e}{=} p) \wedge (\langle p, p' \rangle; (\Gamma \cup \Gamma' \supseteq \Gamma') \stackrel{e}{=} p')$
for all $E, \Gamma, \Gamma' \subseteq \mathsf{Sen}(\Sigma)$ with $\Gamma \cap \Gamma' = \emptyset$

$(P2^{\Sigma})$ $\quad (\forall p, p')(E \xrightarrow{p} \Gamma \cup \Gamma') \wedge (E \xrightarrow{p'} \Gamma \cup \Gamma') \wedge$
$\wedge (p; (\Gamma \cup \Gamma' \supseteq \Gamma) \stackrel{e}{=} p'; (\Gamma \cup \Gamma' \supseteq \Gamma)) \wedge (p; (\Gamma \cup \Gamma' \supseteq \Gamma') \stackrel{e}{=} p'; (\Gamma \cup \Gamma' \supseteq \Gamma')) \Rightarrow p \stackrel{e}{=} p'$
for all $E, \Gamma, \Gamma' \subseteq \mathsf{Sen}(\Sigma)$ with $\Gamma \cap \Gamma' = \emptyset$

The category $\mathsf{Pf}(\Sigma)$ of the Σ-proofs is defined by $|\mathsf{Pf}(\Sigma)| = \mathcal{P}(\mathsf{Sen}(\Sigma))$ and $\mathsf{Pf}(\Sigma)(\Gamma, E) = \{p \in PT^{\Sigma} \mid PT_h^{\Sigma}(p) = \Gamma, PT_c^{\Sigma}(p) = E\}$. The composition of proofs is given by $p; p' = p(PT^{\Sigma}_{\_;\_})p'$ and the monotonicity proofs $\supseteq_{\Gamma, E} : \ \Gamma \to E$ are defined as $PT^{\Sigma}_{\Gamma \supseteq E}$. Notice also that $(PT^{\Sigma})_E = E$. By the last equations above, $\mathcal{P}(\mathsf{Sen}(\Sigma))^{\mathrm{op}} \hookrightarrow \mathsf{Pf}(\Sigma)$ preserves products as each $\Gamma \supseteq E$ gets mapped to $\supseteq_{\Gamma, E}$.

Any signature morphism $\varphi : \ \Sigma \to \Sigma'$ induces a morphism $\overline{\varphi}$ between the theories corresponding to Σ and Σ'. Notice that $\overline{\varphi}$ maps $P1^{\Sigma}$ to $P1^{\Sigma'}$ and $P2^{\Sigma}$ to $P2^{\Sigma'}$ because $\mathsf{Sen}(\varphi)$ is injective. Then we define the functor $\mathsf{Pf}(\varphi)$ as the unique partial algebra homomorphism $PT^{\Sigma} \to PT^{\Sigma'}\restriction_{\overline{\varphi}}$. We have therefore defined a proof system $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf})$, which we will show is the free proof system over $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c)$.

For each signature Σ, let $\eta_{\Sigma} : \ \mathsf{Rl}(\Sigma) \to \mathsf{Pf}(\Sigma)$ map any Σ-rule to its congruence class. We show that the comorphism $(1_{\mathbb{S}ign}, 1_{\mathsf{Sen}}, \eta) : \ (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c) \to (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}, dom, cod)$ is universal.

$$\begin{tikzcd}
(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c) \arrow[rr, "(1_{\mathbb{S}ign}, 1_{\mathsf{Sen}}, \eta)"] \arrow[dr, "(\Phi,\alpha,\gamma)"'] & & (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}, dom, cod) \arrow[dl, "(\Phi,\alpha,\gamma')"] \\
& (\mathbb{S}ign', \mathsf{Sen}', \mathsf{Pf}', dom, cod) &
\end{tikzcd}$$

For each comorphism $(\Phi, \alpha, \gamma) : \ (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c) \to (\mathbb{S}ign', \mathsf{Sen}', \mathsf{Pf}', dom, cod)$, each signature $\Sigma \in |\mathbb{S}ign|$, determines a partial algebra A of the theory of quasi-existence equations defining PT^{Σ} by letting its carrier be $\mathsf{Pf}'(\Phi(\Sigma))$, $A_E = \alpha_{\Sigma}(E)$ for each set E of Σ-sentences, $A_r = \gamma_{\Sigma}(r)$ for each Σ-rule r, A_h, A_c, and $A_{\_;\_}$ respectively, are the canonical extensions of $dom_{\Phi(\Sigma)}$, $cod_{\Phi(\Sigma)}$, and of the composition in $\mathsf{Pf}'(\Phi(\Sigma))$ respectively. Finally, by the universal property of products, we define $A_{\langle\_,\_\rangle}(p_1, p_2)$ to be the unique proof q such that $q; (cod(p_1) \cup cod(p_2) \supseteq cod(p_i)) = p_i$.

Then $\gamma'_{\Sigma} : \ \mathsf{Pf}(\Sigma) \to \mathsf{Pf}'(\Phi(\Sigma))$ given by the unique algebra homomorphism $PT^{\Sigma} \to A$. ∎

REMARK 3.13
For the actual systems of proof rules, the injectivity of the sentence translations comes as a consequence of the injectivity of the signature morphisms. This can be noted easily in the case of **FOL**. Therefore, we cannot have a proof system for **FOL** freely generated from the rules unless we consider its sub-institution determined by all *injective* signature morphisms.

4 Proof-theoretic quantifiers and generalization meta-rule

The paper [23] introduces and discusses proof-theoretic logical connectives such as conjunction, disjunction, falsum, negation, implication, etc., and also universal and existential quantification, as properties of the proof system. In this section we focus on universal quantification.

Lawvere [19] defined quantification as adjoint to substitution. The following was formulated by [23] and defines quantification as adjoint to sentence translation along a class $\mathcal{D}$ of signature morphisms, which typically introduce new constants to serve as quantification 'variables'.

DEFINITION 4.1
Let $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf})$ be a proof system. For any class $\mathcal{D} \subseteq \mathbb{S}ign$ of signature morphisms which is stable under pushouts, i.e. if for any pushout square

$$\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r, "u'"'] & \bullet
\end{tikzcd}$$

$u' \in \mathcal{D}$ whenever $u \in \mathcal{D}$, the proof system *has proof-theoretic universal $\mathcal{D}$-quantification*, if for all signature morphisms $\varphi \in \mathcal{D}$, $\mathsf{Pf}(\varphi)$ have distinguished right adjoints, denoted by $(\forall\varphi)\_$, and which are preserved by proof translations along signature morphisms. This means that there exists a bijective correspondence between $\mathsf{Pf}(\Sigma)(E, (\forall\varphi)E')$ and $\mathsf{Pf}(\Sigma')(\varphi(E), E')$ natural in E and E', such that for each signature pushout with $\varphi \in \mathcal{D}$,

$$\begin{tikzcd} \Sigma \arrow[r, "\theta"] \arrow[d, "\varphi"] & \Sigma_1 \arrow[d, "\varphi_1"] \\ \Sigma' \arrow[r, "\theta'"'] & \Sigma'_1 \end{tikzcd}$$

both squares below commute:

$$\begin{tikzcd} \mathsf{Pf}(\Sigma) \arrow[r, "\mathsf{Pf}(\theta)"] & \mathsf{Pf}(\Sigma_1) \\ \mathsf{Pf}(\Sigma') \arrow[u, "(\forall\varphi)"] \arrow[r, "\mathsf{Pf}(\theta')"'] & \mathsf{Pf}(\Sigma'_1) \arrow[u, "(\forall\varphi_1)"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Pf}(\Sigma)(E,(\forall\varphi)E') \arrow[r, "\sim"] \arrow[d, "\mathsf{Pf}(\theta)"'] & \mathsf{Pf}(\Sigma')(\varphi(E),E') \arrow[d, "\mathsf{Pf}(\theta')"] \\ \mathsf{Pf}(\Sigma_1)(\theta(E),(\forall\varphi_1)\theta'(E')) \arrow[r, "\sim"] & \mathsf{Pf}(\Sigma'_1)(\varphi_1(\theta(E)),\theta'(E')) \end{tikzcd}$$

Such a proof system with universal quantification may be denoted by $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, \forall)$.

REMARK 4.2
In categorical terms, the commutativity of these squares just says that the pair $\langle \mathsf{Pf}(\theta),\ \mathsf{Pf}(\theta')\rangle$ is a morphism of adjunctions.

REMARK 4.3
Definition 4.1 refines the so-called 'Generalization' meta-rule from classical logic

$$E \vdash (\forall\chi)\rho \ \text{ iff } \ \chi(E) \vdash \rho$$

for any $\chi \in \mathcal{D}$. This Generalization' meta-rule means that the underlying entailment system has universal $\mathcal{D}$-quantification (in the sense of Definition 4.1).

REMARK 4.4
Proof-theoretic existential quantification can be handled similarly to universal quantification, with $(\exists\varphi)_{\_}$ appearing as a left adjoint to $\mathsf{Pf}(\varphi)$.

Quantification can be added freely to proof systems already admitting a syntax for quantifiers, a condition which is captured by the following concept:

DEFINITION 4.5
A *sentence system with pre-quantifiers* $(\mathbb{S}ign, \mathsf{Sen}, \mathcal{D}, Q)$ consists, like institutions or proof systems, of a signature category $\mathbb{S}ign$ and a sentence functor Sen, but also marks a subcategory $\mathcal{D}$ of signature morphisms which is stable under pushouts, and a functor $Q: \ \mathcal{D} \to \mathbb{S}et^{\mathrm{op}}$ such that for each pushout of signature morphisms

$$\begin{tikzcd} \Sigma \arrow[r, "\theta"] \arrow[d, "\varphi"'] & \Sigma_1 \arrow[d, "\varphi_1"] \\ \Sigma' \arrow[r, "\theta'"'] & \Sigma'_1 \end{tikzcd}$$

with $\varphi \in \mathcal{D}$, the square below commutes:

$$\begin{tikzcd}
\mathsf{Sen}(\Sigma) \arrow[r, "\mathsf{Sen}(\theta)"] & \mathsf{Sen}(\Sigma_1) \\
\mathsf{Sen}(\Sigma') \arrow[u, "Q(\varphi)"] \arrow[r, "\mathsf{Sen}(\theta')"'] & \mathsf{Sen}(\Sigma'_1) \arrow[u, "Q(\varphi_1)"']
\end{tikzcd}$$

An *institution or a proof system has pre-quantifiers* when its underlying sentence system has pre-quantifiers.

Note that the part of institutions with pre-quantifiers without the functor Q are called ‘institutions with signature variables’ in [28].

EXAMPLE 4.6
A sentence system with pre-quantifiers for **FOL**, corresponding to **FOL** universal quantification, consists of $\mathbb{S}ign^{\mathbf{FOL}}$ as signature category, $\mathsf{Sen}^{\mathbf{FOL}}$ as sentence functor, $\mathcal{D}$ all signature extensions with a finite number of constants, and $Q(\chi)(\rho) = (\forall\chi)\rho$ for each $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$ and each $\rho \in \mathsf{Sen}^{\mathbf{FOL}}(\Sigma')$.

Another sentence system with pre-quantifiers, this time corresponding to **FOL** existential quantification, may be given by $Q(\chi)(\rho) = (\exists\chi)\rho$.

DEFINITION 4.7
A *comorphism of sentence systems with pre-quantifiers* $(\Phi, \alpha) : (\mathbb{S}ign, \mathsf{Sen}, \mathcal{D}, Q) \to (\mathbb{S}ign', \mathsf{Sen}', \mathcal{D}', Q')$ consists of a functor $\Phi : \mathbb{S}ign \to \mathbb{S}ign'$, and a natural transformation $\alpha : \mathsf{Sen} \Rightarrow \Phi; \mathsf{Sen}'$ which is also a natural transformation $Q \Rightarrow \Phi; Q'$. A *comorphism of institution/proof system with pre-quantifiers* combines an institution/proof system comorphism with a comorphism between the underlying sentence systems with pre-quantifiers.

THEOREM 4.8
The forgetful functor from proof systems with universal quantification to proof systems with pre-quantifiers has a left adjoint.

PROOF. Let $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, Q)$ be a proof system with pre-quantifiers. This defines the following (one-sorted) quasi-existence equational theory:

- theories (Σ, E) and pairs of theories $(\Sigma, E) \supseteq (\Sigma, E')$ for all $E' \supseteq E$ as total constants,
- h and c as total unary operation symbols,
- all signature morphisms $\varphi \in \mathbb{S}ign$ as partial unary operation symbols, and
- $\_;\_$ and $\langle\_,\_\rangle$ as binary partial operation symbols,

and of the following set of quasi-existence equations:

- (S),(M1-3),(C1-3),(P1-2) as in the proof of Theorem 3.12 but in a version replacing E’s by (Σ, E)’s, and

$$\begin{array}{ll}
(FS) & (\forall p)(\Sigma, E) \xrightarrow{p} (\Sigma, E') \Rightarrow (\Sigma', \varphi(E)) \xrightarrow{\varphi(p)} (\Sigma', \varphi(E')) \\
(FM) & \varphi((\Sigma, E') \supseteq (\Sigma, E)) \stackrel{e}{=} (\Sigma', \varphi(E')) \supseteq (\Sigma', \varphi(E)) \\
(FC) & (\forall p, p')((\Sigma, E) \xrightarrow{p} (\Sigma, E') \wedge ((\Sigma, E') \xrightarrow{p'} (\Sigma, E'') \Rightarrow \varphi(p; p') \stackrel{e}{=} \varphi(p); \varphi(p') \\
(FF) & (\forall p)\varphi'(\varphi(p)) \stackrel{e}{=} (\varphi; \varphi')(p) \\
(FI) & (\forall p)1_\Sigma(p) \stackrel{e}{=} p
\end{array}$$

for all signature morphisms $\varphi : \ \Sigma \to \Sigma'$ and $\varphi' : \ \Sigma' \to \Sigma''$, all sets of sentences $E, E', E'' \subseteq \mathsf{Sen}(\Sigma)$, and where $(\Sigma, E) \xrightarrow{p} (\Sigma, \Gamma)$ abbreviate $(h(p) \stackrel{e}{=} (\Sigma, E)) \wedge (c(p) \stackrel{e}{=} (\Sigma, \Gamma))$

The proof system determines canonically a partial algebra P of this quasi-existence equational theory with its underlying set being the disjoint union of proof categories $\biguplus_{\Sigma \in |\mathbb{S}ign|} \mathsf{Pf}(\Sigma)$, and interpreting the operation symbols in the obvious way.

Now, we extend the above quasi-existence equational theory with

- partial unary operations $[\forall\varphi]$ for all signature morphism $\varphi \in \mathcal{D}$,
- total constants $\eta^{\varphi}_{(\Sigma,E)}$ and $\epsilon^{\varphi}_{(\Sigma,E)}$ for each theory (Σ, E) and each signature morphism $\varphi \in \mathcal{D}$,

and with the following sentences:

$$
\begin{array}{ll}
(QS_0) & [\forall\varphi](E') \stackrel{e}{=} Q(E') \\
(QS) & (\forall p)(\Sigma', \Gamma') \xrightarrow{p} (\Sigma', E') \Rightarrow (\Sigma, [\forall\varphi]\Gamma') \xrightarrow{[\forall\varphi]p} (\Sigma, [\forall\varphi]E') \\
(QC) & (\forall p, p')((\Sigma', \Gamma') \xrightarrow{p} (\Sigma', E')) \wedge ((\Sigma', E') \xrightarrow{p'} (\Sigma', E'')) \Rightarrow [\forall\varphi](p; p') \stackrel{e}{=} [\forall\varphi]p; [\forall\varphi]p' \\
(QF) & (\forall p)((\Sigma', \Gamma') \xrightarrow{p} (\Sigma', E')) \Rightarrow \theta([\forall\varphi]p) \stackrel{e}{=} [\forall\varphi_1](\theta'(p)) \\
(ET_0) & (\Sigma, E) \xrightarrow{\eta^{\varphi}_{(\Sigma,E)}} (\Sigma, [\forall\varphi]\varphi(E)) \\
(ET_1) & (\forall p)(\Sigma, \Gamma) \xrightarrow{p} (\Sigma, E) \Rightarrow \eta^{\varphi}_{(\Sigma,\Gamma)}; [\forall\varphi]\varphi(p) \stackrel{e}{=} p; \eta^{\varphi}_{(\Sigma,E)} \\
(EP_0) & (\forall p)(\Sigma', \varphi([\forall\varphi]E')) \xrightarrow{\epsilon^{\varphi}_{(\Sigma',E')}} (\Sigma', E') \\
(EP_1) & (\forall p')(\Sigma', \Gamma') \xrightarrow{p'} (\Sigma', E') \Rightarrow \epsilon^{\varphi}_{(\Sigma',\Gamma')}; p' \stackrel{e}{=} \varphi([\forall\varphi]p'); \epsilon^{\varphi}_{(\Sigma',E')} \\
(TF) & \varphi(\eta^{\varphi}_{(\Sigma,E)}); \epsilon^{\varphi}_{(\Sigma',\varphi(E))} \stackrel{e}{=} (\Sigma', \varphi(E)) \\
(TQ) & \eta^{\varphi}_{(\Sigma,[\forall\varphi]E')}; [\forall\varphi]\epsilon^{\varphi}_{(\Sigma',E')} \stackrel{e}{=} (\Sigma, [\forall\varphi]E') \\
(I) & (\theta(\eta^{\varphi}_{(\Sigma,E)}) \stackrel{e}{=} \eta^{\varphi_1}_{(\Sigma_1,\theta(E))}) \wedge (\theta'(\epsilon^{\varphi}_{(\Sigma',E')}) \stackrel{e}{=} \epsilon^{\varphi_1}_{(\Sigma'_1,\theta'(E'))})
\end{array}
$$

for all signature morphisms $(\varphi : \ \Sigma \to \Sigma') \in \mathcal{D}$, $\Gamma', E', E'' \subseteq \mathsf{Sen}(\Sigma')$, $E \subseteq \mathsf{Sen}(\Sigma)$, and all signature pushouts

$$
\begin{tikzcd}
\Sigma \arrow[r, "\theta"] \arrow[d, "\varphi"'] & \Sigma_1 \arrow[d, "\varphi_1"] \\
\Sigma' \arrow[r, "\theta'"'] & \Sigma'_1
\end{tikzcd}
$$

Let $\overline{P}$ be the free extension of P along the extension δ of quasi-existence equational theories defined above. Let $\zeta : \ P \to \overline{P}\restriction_\delta$ denote the universal partial algebra homomorphism. For each signature $\Sigma \in |\mathbb{S}ign|$, by letting $\overline{\mathsf{Pf}}(\Sigma)(E, E') = \{p \in \overline{P} \mid (\Sigma, E) \xrightarrow{p} (\Sigma, E')\}$ we get a proof system with universal quantification $(\mathbb{S}ign, \mathsf{Sen}, \overline{\mathsf{Pf}}, \mathcal{D}, \forall)$ and a comorphism of proof systems with pre-quantifiers $(1, 1, \omega) : \ (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, Q) \to (\mathbb{S}ign, \mathsf{Sen}, \overline{\mathsf{Pf}}, \mathcal{D}, \forall)$ where $\omega_\Sigma(p) = \zeta(p)$.

Any comorphism of proof systems with pre-quantifiers $(\Phi, \alpha, \gamma) : \ (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, Q) \to (\mathbb{S}ign', \mathsf{Sen}', \mathsf{Pf}', \mathcal{D}', \forall)$ to a proof system with universal quantifiers determines canonically a partial algebra homomorphism $\gamma' : \ P \to P'\restriction_\delta$ mapping each Σ-proof $p : \ \Gamma \to E$ to the $\Phi(\Sigma)$-proof $\gamma_\Sigma(p) : \ \alpha_\Sigma(\Gamma) \to \alpha_\Sigma(E)$, where P' is the partial algebra of the extended quasi-equational theory with carrier $\biguplus_{\Sigma' \in |\mathbb{S}ign'|} \mathsf{Pf}'(\Sigma')$.

Then the unique partial algebra homomorphism $\overline{\gamma'} : \ \overline{P} \to P'$ such that $\zeta; \overline{\gamma'}\restriction_\delta = \gamma'$ determines back a comorphism of proof systems with universal quantifiers $(\Phi, \alpha, \overline{\gamma})$ such that

$$
\begin{tikzcd}
(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, Q) \arrow[rr, "{(1,1,\omega)}"] \arrow[dr, "{(\Phi,\alpha,\gamma)}"'] & & (\mathbb{S}ign, \mathsf{Sen}, \overline{\mathsf{Pf}}, \mathcal{D}, \forall) \arrow[dl, "{(\Phi,\alpha,\overline{\gamma})}"] \\
& (\mathbb{S}ign', \mathsf{Sen}', \mathsf{Pf}', \mathcal{D}', \forall) &
\end{tikzcd}
$$

■

REMARK 4.9
A variant of Theorem 4.8 may generate the universally quantified sentences freely, thus eliminating the need of the pre-quantifier structure. The reader is invited to explore the details of this idea. Here we have preferred the approach based on pre-quantifiers mainly because our intention is to add proof system structures to institutions without having to extend their sentences.

5 Compactness

In this section we develop some general compactness results. As a matter of notation, in the diagrams in the category of proofs, unlabelled arrows will denote monotonicity proofs.

DEFINITION 5.1
A proof rule r (in a system of proof rules $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c)$) is *finitary* when both the hypothesis $h_\Sigma(r)$ and the conclusion $c_\Sigma(r)$ are finite for each signature Σ.

A proof (in a proof system)

- is *finitary* when it is finitary as a proof rule (via the forgetful functor from proof systems to systems of proof rules), and
- is *compact* when it can be represented as $\langle \supseteq_{E,E_1}, \supseteq_{E,E_2'}; q\rangle$ with q finitary.

$$
\begin{tikzcd}
E_1 & E_1 \uplus E_2 \arrow[l] \arrow[rr] & & E_2 \\
& E \arrow[ul] \arrow[u, "{\langle \supseteq_{E,E_1}, \supseteq_{E,E_2'}; q\rangle}"'] \arrow[rr] & & E_2' \arrow[u, "q"']
\end{tikzcd}
$$

A proof system is *compact* when each proof is compact.

REMARK 5.2
Notice that any compact proof $E \to \Gamma$ with Γ finite can be written as a composition between a monotonicity proof and a finitary proof. Because of the trivial nature of monotonicity proofs in actual proof systems, one can see that in any compact proof system each proof of a finite set of sentences is essentially finitary.

The concept of compact proof system introduced by Definition 5.1 is stronger than the usual compactness in classical logic which is a compactness property of the entailment system rather than of the proof system. Otherwise said, while classical compactness can be understood in the sense that any provable sentence *admits* a finitary proof, our compactness says that any proof of a sentence *is* (essentially) finitary.

PROPOSITION 5.3
For any proof system $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf})$, the collection of its compact proofs form a (proof) sub-system, with $C(\mathsf{Pf})$ denoting its proof functor.

PROOF. We have only to show that compact proofs form a sub-category of all proofs, and that this sub-category creates (binary) products of disjoint sets of sentences, other necessary facts like the preservation of compactness by translations along signature morphisms being obvious.

1. *Subcategory.* Notice that each identity proof is trivially compact. Consider proofs $\langle \supseteq_{\Gamma,E_1}, \supseteq_{\Gamma,E'_2}; q\rangle : \Gamma \to E_1 \uplus E_2$ with $E'_2 \xrightarrow{q} E_2$ finitary and $\langle \supseteq_{E,\Gamma_1}, \supseteq_{E,\Gamma'_2}; r\rangle : E \to \Gamma_1 \uplus \Gamma_2$ with $\Gamma'_2 \xrightarrow{r} \Gamma_2$ finitary. We have to show that their composition is compact.

$$\begin{tikzcd}
 & \Gamma_1 & & E_1 \\
E \arrow[ur] \arrow[r, "{\langle \supseteq_{E,\Gamma_1}, \supseteq_{E,\Gamma'_2}; r\rangle}"'] \arrow[d] & \Gamma_1 \uplus \Gamma_2 \arrow[u] \arrow[d] \arrow[r, equal] & \Gamma \arrow[ur] \arrow[r, "{\langle \supseteq_{\Gamma,E_1}, \supseteq_{\Gamma,E'_2}; q\rangle}"'] \arrow[d] & E_1 \uplus E_2 \arrow[u] \arrow[d] \\
\Gamma'_2 \arrow[r, "r"'] & \Gamma_2 & E'_2 \arrow[r, "q"'] & E_2
\end{tikzcd}$$

We know that $E'_2 \subseteq \Gamma = \Gamma_1 \uplus \Gamma_2$, hence $E'_2 = (E'_2 \cap \Gamma_1) \uplus (E'_2 \cap \Gamma_2)$. We show that without any loss of generality we may assume that $E'_2 \subseteq \Gamma_2$. If $E'_2 \not\subseteq \Gamma_2$, then because $\langle \supseteq_{E,\Gamma_1}, \supseteq_{E,\Gamma'_2}; r\rangle = \langle \supseteq_{E,\Gamma_1 \setminus E'_2}, \supseteq_{E_1,\Gamma'_2 \uplus (\Gamma_1 \cap E'_2)}; (1_{E'_2 \cap \Gamma_1} \uplus r)\rangle$ we may replace Γ_2 by $E'_2 \cup \Gamma_2$. The above equality can be established by projecting to Γ_1 and Γ_2 and by using the uniqueness part from the universal property of the (monotonicity proofs) projections $\Gamma_1 \longleftarrow \Gamma_1 \uplus \Gamma_2 \longrightarrow \Gamma_2$.

$$\begin{tikzcd}
 & \Gamma_1 \cap E'_2 & & \\
\Gamma_1 \arrow[ur] \arrow[d] & \Gamma_1 \uplus \Gamma_2 = \Gamma = (\Gamma_1 \setminus E'_2) \uplus (E'_2 \cup \Gamma_2) \arrow[l] \arrow[u] \arrow[r] \arrow[dl] & (E'_2 \cup \Gamma_2) = (\Gamma_1 \cap E'_2) \uplus \Gamma_2 \arrow[ul] \arrow[r] & \Gamma_2 \\
(\Gamma_1 \setminus E'_2) & E \arrow[l] \arrow[ul] \arrow[u, "x"'] \arrow[r, "{\supseteq_{E,\Gamma'_2 \uplus (\Gamma_1 \cap E'_2)} = \langle \supseteq_{E,\Gamma'_2}, \supseteq_{E,\Gamma_1 \cap E'_2}\rangle}"'] & (\Gamma_1 \cap E'_2) \uplus \Gamma'_2 \arrow[uul] \arrow[u, "(1_{\Gamma_1 \cap E'_2} \uplus r)"'] \arrow[r] & \Gamma'_2 \arrow[u, "r"]
\end{tikzcd}$$

Then, assuming that $E'_2 \subseteq \Gamma_2$, we have that $\langle \supseteq_{E,\Gamma_1}, \supseteq_{E,\Gamma'_2}; r\rangle; \langle \supseteq_{\Gamma,E_1}, \supseteq_{\Gamma,E'_2}; q\rangle =$ $\langle \supseteq_{E,E_1 \cap \Gamma_1}, \supseteq_{E,\Gamma'_2}; r; \langle \supseteq_{\Gamma_2,E'_2}; q, \supseteq_{\Gamma_2,E_1 \cap \Gamma_2}\rangle\rangle$. Notice that $\langle \supseteq_{E,E_1 \cap \Gamma_1}, \supseteq_{E,\Gamma'_2}; r; \langle \supseteq_{\Gamma_2,E'_2}; q, \supseteq_{\Gamma_2,E_1 \cap \Gamma_2}\rangle\rangle$ is compact.

This equality can be established by projecting to E_2, $E_1 \cap \Gamma_1$ and $E_1 \cap \Gamma_2$ and by using the uniqueness part from the universal property of the (monotonicity proof) projections of the product (in $\mathsf{Pf}(\Sigma)$) $E_2 \uplus (E_1 \cap \Gamma_1) \uplus (E_1 \cap \Gamma_2)$.

$$\begin{tikzcd}
E_1 \arrow[d] & (E_1 \cap \Gamma_1) \uplus E_2 \uplus (E_1 \cap \Gamma_2) \arrow[l] \arrow[rr] \arrow[dl] \arrow[dr] & & E_2 \\
E_1 \cap \Gamma_1 & & E_1 \cap \Gamma_2 & E'_2 \arrow[u, "q"] \\
\Gamma_1 \arrow[u] & E \arrow[l] \arrow[uu, "y"] \arrow[r] & \Gamma'_2 \arrow[r, "r"'] & \Gamma_2 \arrow[u] \arrow[ul]
\end{tikzcd}$$

2. *Direct products of disjoint sets.* Assume compact proofs $\langle \supseteq_{E,E_1}, \supseteq_{E,E'_2}; q\rangle$ and $\langle \supseteq_{E,\Gamma_1}, \supseteq_{E,\Gamma'_2}; r\rangle$ such that $(E_1 \uplus E_2) \cap (\Gamma_1 \uplus \Gamma_2) = \emptyset$. The fact that $\langle\langle \supseteq_{E,E_1}, \supseteq_{E,E'_2}; q\rangle, \langle \supseteq_{E,\Gamma_1}, \supseteq_{E,\Gamma'_2}; r\rangle\rangle : \ E \to E_1 \uplus E_2 \uplus \Gamma_1 \uplus \Gamma_2$ is compact too follows immediately from the equality
$\langle\langle \supseteq_{E,E_1}, \supseteq_{E,E'_2}; q\rangle, \langle \supseteq_{E,\Gamma_1}, \supseteq_{E,\Gamma'_2}; r\rangle\rangle = \langle \supseteq_{E,E_1 \uplus \Gamma_1}, \supseteq_{E,E'_2 \cup \Gamma'_2}; \langle \supseteq_{E'_2 \cup \Gamma'_2, E'_2}; q, \supseteq_{E'_2 \cup \Gamma'_2, \Gamma'_2}; r\rangle\rangle$. ■

COROLLARY 5.4
The proof system freely generated by a system of finitary proof rules is compact.

PROOF. Consider proof system $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf})$ generated freely by a system of finitary proof rules $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c)$, with $(1_{\mathbb{S}ign}, 1_{\mathsf{Sen}}, \eta)$ universal arrow.

By Proposition 5.3 let $(\mathbb{S}ign, \mathsf{Sen}, C(\mathsf{Pf}))$ be the compact proof (sub-)system of $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf})$. Because each proof rule of $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c)$ is finitary, it means that $\eta_\Sigma(\mathsf{Rl}(\Sigma)) \subseteq C(\mathsf{Pf})(\Sigma)$ for each signature Σ, hence $(1_{\mathbb{S}ign}, 1_{\mathsf{Sen}}, \eta)$ is a comorphism of systems of proof rules $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c) \to (\mathbb{S}ign, \mathsf{Sen}, C(\mathsf{Pf}), dom, cod)$.
By the universal property of $(1_{\mathbb{S}ign}, 1_{\mathsf{Sen}}, \eta) : \ (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c) \to (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}, dom, cod)$ there exists an unique comorphism of proof systems $(1, 1, \gamma) : \ (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}) \to (\mathbb{S}ign, \mathsf{Sen}, C(\mathsf{Pf}))$ such that the following diagram commutes:

$$\begin{tikzcd}
(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c) \arrow[r, "{(1_{\mathbb{S}ign}, 1_{\mathsf{Sen}}, \eta)}"] \arrow[dr, "{(1_{\mathbb{S}ign}, 1_{\mathsf{Sen}}, \eta)}"'] & (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}, dom, cod) \arrow[d, "{(1,1,\gamma)}"] \\
& (\mathbb{S}ign, \mathsf{Sen}, C(\mathsf{Pf}), dom, cod)
\end{tikzcd}$$

Let $(1, 1, \gamma')$ be the sub-system comorphism $(\mathbb{S}ign, \mathsf{Sen}, C(\mathsf{Pf})) \to (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf})$, which also makes the above triangle commute. By the uniqueness part of the universal property for the the free proof system, we get that $\gamma; \gamma' = 1$, and because γ' are inclusions, we obtain that $C(\mathsf{Pf}) = \mathsf{Pf}$, which means that each proof of $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf})$ is compact. ■

COROLLARY 5.5
The proof system with universal quantification freely generated by a compact proof system with pre-quantifiers is compact too.

PROOF. This follows an argument very similar to that used for the proof of Corollary 5.4 by taking the compact proof (sub-)system $(\mathbb{S}ign, \mathsf{Sen}, C(\overline{\mathsf{Pf}}))$ (cf. Proposition 5.3) of the free proof system with universal quantification $(\mathbb{S}ign, \mathsf{Sen}, \overline{\mathsf{Pf}})$, and by noticing that proof system comorphisms preserve compact proofs which means that, if we assume that $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf})$ is compact, then the universal comorphism $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}) \to (\mathbb{S}ign, \mathsf{Sen}, \overline{\mathsf{Pf}})$ goes in fact to $(\mathbb{S}ign, \mathsf{Sen}, C(\overline{\mathsf{Pf}}))$. ■

6 Soundness

In this section we develop some general soundness results.

DEFINITION 6.1
An *institution with proofs* $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Pf})$ puts together an institution $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Mod}, \models)$ and a proof system $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf})$. Similarly, an *institution with proof rules* $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Mod}, \mathsf{Rl}, h, c)$ puts together an institution and a system of proof rules.

DEFINITION 6.2
An institution with proof rules $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Rl}, h, c)$ is *sound* when for each rule $r \in \mathsf{Rl}(\Sigma)$, $h_\Sigma(r) \models c_\Sigma(r)$. An institution with proofs is *sound* if it is sound when regarded as an institution with proof rules (i.e. $E \vdash \Gamma$ implies $E \models \Gamma$), and it is *complete* when $E \models_\Sigma \Gamma$ implies $E \vdash_\Sigma \Gamma$ for all sets $E, \Gamma \subseteq \mathsf{Sen}(\Sigma)$.

The soundness of the actual institutions with proofs follows automatically from the soundness of the generator institution with proof rules.

PROPOSITION 6.3
The free institution with proofs $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Pf})$ generated by any sound institution with proof rules $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Rl}, h, c)$ is sound too.

PROOF. Because $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Mod}, \mathsf{Rl}, h, c)$ is sound we consider the canonical comorphism of systems of proof rules $(1, 1, \gamma) : \ (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c) \to (\mathbb{S}ign, \mathsf{Sen}, \models, dom, cod)$ to the institution with semantic consequence as proofs.

$$\begin{tikzcd}
(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Rl}, h, c) \arrow[rr, "(1_{\mathbb{S}ign}, 1_{\mathsf{Sen}}, \eta)"] \arrow[dr, "(1_{\mathbb{S}ign}, 1_{\mathsf{Sen}}, \gamma)"'] & & (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}) \arrow[dl, "(1_{\mathbb{S}ign}, 1_{\mathsf{Sen}}, \gamma')"] \\
& (\mathbb{S}ign, \mathsf{Sen}, \models) &
\end{tikzcd}$$

By the universal property of the free proof system $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf})$, $(1, 1, \gamma)$ can be extended to a comorphism of proof systems $(1, 1, \gamma') : \ (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}) \to (\mathbb{S}ign, \mathsf{Sen}, \models)$, which gives the soundness of $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Pf})$. ∎

Recall [30, 8] that an institution *has semantic universal $\mathcal{D}$-quantification* for a class $\mathcal{D} \subseteq \mathbb{S}ign$ of signature morphisms when for each $(\chi : \ \Sigma \to \Sigma') \in \mathcal{D}$ and each $\rho \in \mathsf{Sen}(\Sigma')$ there exists a sentence in $\mathsf{Sen}(\Sigma)$, denoted by $(\forall\chi)\rho$, such that

$$M \models (\forall\chi)\rho \text{ iff } M' \models \rho \text{ for each } \chi\text{-expansion } M' \text{ of } M$$

for each Σ-model M. Semantic existential quantification can be defined similarly. This 'institution-independent' concept of semantic quantification captures ordinary quantification of the actual institutions, for example **FOL** has $\mathcal{D}$-quantification for $\mathcal{D}$ the class of signature extensions with a finite number of constants, while in then case of second order logic then class $\mathcal{D}$ consists of (finite) extensions with any relation and any operation symbols, and in the case of **PA**, $\mathcal{D}$ is the class of signature extensions with a finite number of total constants.

PROPOSITION 6.4
Let $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Pf})$ be any sound institution with proofs and with semantic universal $\mathcal{D}$-quantifiers. Then the institution with proofs having proof-theoretic universal $\mathcal{D}$-quantifiers which is freely generated by $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Pf})$, where the pre-quantifiers are given by the semantic universal quantifiers, is sound too.

PROOF. Notice that the semantic entailment system $(\mathbb{S}ign, \mathsf{Sen}, \models, \mathcal{D}, \forall)$ is a proof system with universal quantification. By the soundness of $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Mod}, \models, \mathsf{Pf})$ we get a comorphism

$(1, 1, \gamma) :\ (\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, \forall) \to (\mathbb{S}ign, \mathsf{Sen}, \models, \mathcal{D}, \forall)$ of proof systems with pre-quantifiers.

$$\begin{tikzcd}
(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Pf}, \mathcal{D}, \forall) \arrow[rr, "(1_{\mathbb{S}ign}, 1_{\mathsf{Sen}}, \omega)"] \arrow[dr, "(1_{\mathbb{S}ign}, 1_{\mathsf{Sen}}, \gamma)"'] & & (\mathbb{S}ign, \mathsf{Sen}, \overline{\mathsf{Pf}}, \mathcal{D}, \forall) \arrow[dl, "(1_{\mathbb{S}ign}, 1_{\mathsf{Sen}}, \overline{\gamma})"] \\
& (\mathbb{S}ign, \mathsf{Sen}, \models, \mathcal{D}, \forall) &
\end{tikzcd}$$

By the universal property of $(\mathbb{S}ign, \mathsf{Sen}, \overline{\mathsf{Pf}}, \mathcal{D}, \forall)$ we get a comorphism $(1, 1, \overline{\gamma})$ of proof systems with universal quantification, giving the soundness of $(\mathbb{S}ign, \mathsf{Sen}, \mathsf{Mod}, \models, \overline{\mathsf{Pf}})$. ∎

7 Conclusions and future research

We have introduced a concept of proof rule for institutions, by specifiying an algebra of proofs we have showed that any system of proof rules generates freely a proof system in the sense of [23], and we have argued that actual proof systems are freely generated by their presentations as systems of proof rules. We have also showed that proof-theoretic quantification can be freely generated over any proof system which admits a syntax for quantifiers. This abstracts the Generalization meta-rule from classical logic to any institution with proof-theoretic structure. These universal constructions have provided

- a compactness theorem for proof systems freely generated by systems of finitary proof rules,
- automatic lifting of compactness to free proof systems with universal quantifiers,
- an automatic lifting of soundness from proof rules to (free) proof systems, and
- by assuming corresponding semantic quantifiers, further (free) institutions with proofs having proof-theoretic universal quantification.

Other logical meta-rules should be considered in the same way as Generalization has been treated here, i.e. as logical properties of the proof system rather than generating proof rules. However it is important that they are expressed in such a way that they can be coded as partial algebra quasi-existence equations (in the style of the proofs of Theorems 3.12 and 4.8), otherwise the existence of proof systems satisfying such meta-rules is lost.

Future research may include finding answers to a number of interesting problems and research directions.

1. Completeness results in an institutional framework. At the time of writing this paper, this has already received an elegant solution in [24] in the sense that completeness can be transported from a proof system of the 'atomic' sub-institution to the main institution by considering a certain system of proof rules for $\neg, \vee, \forall$ formulated in an institution-independent style.
2. When is proof-theoretic quantification 'orthogonal' to the logical connectives? In other words, for example, given a proof system with implication (see [23]), does the free proof system with universal quantification still have implication?
3. Proof systems for institutional logic might provide an adequate framework for foundations of verification methodologies of formal specifications as an alternative to the existing type theoretic frameworks. This topic is important for software engineering because the complexity of proof scores is at least one order of magnitude bigger than that of writing formal specifications. And in spite of this situation, while formal specifications have solid mathematical foundations, the 'science' of proof scores lack them, at least from an algebraic specification culture perspective.

4. The study of important logical properties such as interpolation and definability in the proof theoretic setting introduced by this paper. Some basic definitions have already appeared in [23].
5. The study of reduction behaviour of proofs. In this respect, the work [1] seems to contain ideas and concepts which can be naturally transported to the proof theoretic framework propsed by our work.

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Received 1 March 2005