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Grothendieck Institutions

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Abstract. We extend indexed categories, fibred categories, and Grothendieck constructions to institutions. We show that the 2-category of institutions admits Grothendieck constructions (in a general 2-categorical sense) and that any split fibred institution is equivalent to a Grothendieck institution of an indexed institution.

We use Grothendieck institutions as the underlying mathematical structure for the semantics of multi-paradigm (heterogenous) algebraic specification. We recuperate the so-called 'extra theory morphisms' as ordinary theory morphisms in a Grothendieck institution. We investigate the basic mathematical properties of Grothendieck institutions, such as theory colimits, liberality (free constructions), exactness (model amalgamation), and inclusion systems by 'globalisation' from the 'local' level of the indexed institution to the level of the Grothendieck institution.

1. Introduction

Multi-paradigm (heterogenous) logical specification or programming languages admit institution semantics in which each paradigm has an underlying institution and paradigm embedding formally corresponds to institution morphism. This leads to a concept of *indexed institution* which generalises indexed categories of (Paré and Schumacher, 1978; Tarlecki et al., 1991). Semantics of multi-paradigm specification languages requires the extension of the institution concepts across indexed institutions; this can be naturally achieved by an extension of the Grothendieck construction for indexed categories to indexed institutions, which this leads to the concept of *Grothendieck institution*. We prove that the 2-category of institutions admits internal Grothendieck constructions abstractly expressed as special lax colimits. In a fibration framework, Grothendieck institutions can be formalised as *fibred institutions*, we develop here this concept rather briefly, and show that Grothendieck institutions are categorically equivalent to split fibred institutions by extending a classical result by Bénabou.

The new algebraic specification language CafeOBJ (Diaconescu and Futatsugi, 1998) provides a good practical example for the use of Grothendieck or fibred institutions (Diaconescu and Futatsugi, 2002). In fact, the research on Grothendieck institutions is part of the research project on the logical foundations of CafeOBJ. The semantics of CafeOBJ is based on the indexed institution resulting from the various combinations of the basic CafeOBJ paradigms. This is illustrated by the following so-called 'CafeOBJ cube' (consider only the full arrows):

where the nodes represent institutions and the arrows represent institution morphisms. The institution underlying CafeOBJ is obtained as the Grothendieck institution of the CafeOBJ cube, which is a lax colimit of the CafeOBJ cube in the 2-category of institutions.

The work of this paper can be regarded as a step forward from (Diaconescu, 1998), Grothendieck institutions providing a higher conceptual framework for the so-called 'extra theory morphisms'.

We show that extra theory morphisms of (Diaconescu, 1998) can be regarded as ordinary theory morphisms in a Grothendieck institution. In this way, we come back to the globalisation of institutional properties studied in (Diaconescu, 1998) from the new higher conceptual perspective of the Grothendieck (fibred) institutions. In this paper we extend the main globalisation results of (Diaconescu, 1998) (obtained there in sufficient form) to necessary and sufficient conditions. These include theory colimits, liberality, exactness, and inclusion systems.

Theory colimits. Module expressions in algebraic languages in the Clear-OBJ tradition are evaluated as colimits of theories (Goguen and Burstall, 1992). The problem of existence of theory colimits exhibits very clearly the conceptual power of Grothendieck institutions, which enable a very compact proof (contrasting to the rather complex similar proof of (Diaconescu, 1998)) by using important results from indexed category theory and institution theory.

Liberality. Liberality (Goguen and Burstall, 1992; Tarlecki, 1986) is a basic desirable property expressing the possibility of free constructions generalising the principle of 'initial algebra semantics' which underlies the tight semantics of algebraic languages, including semantics for parameterised modules (Diaconescu et al., 1993). Here we give a necessary and sufficient condition for the liberality of a Grothendieck institution which extends a similar result of (Diaconescu, 1998).

Exactness. Exactness expresses the possibility of amalgamation of consistent models (or 'implementations', in a more application oriented jargon) for different specification modules (for more details see (Diaconescu et al., 1993)) and is a necessary technical condition on the underlying logic for good semantic properties of the module system for a specification language. A set of necessary and sufficient conditions for the globalisation of exactness was the main conjecture of (Diaconescu, 1998), in this paper we solve this problem within the framework of Grothendieck institutions.

Inclusions. Theory inclusions model mathematically the concept of module import (see (Diaconescu et al., 1993)), which is the most fundamental structuring operation for specification languages. *Inclusion systems* were first introduced in (Diaconescu et al., 1993) as the underlying categorical structure of an institution-independent module algebra. They were further studied and their definition simplified in (Căzănescu and Roşu, 1997). Inclusion systems are related to the better established concept of factorisation systems, but they capture the uniqueness property of inclusions (such as set-theoretic inclusions). Here we extend the construction of inclusion systems for extra theory morphisms of (Diaconescu, 1998) to Grothendieck institutions.

2. Preliminaries

2.1. Categories

This work assumes some familiarity with category theory (including 2-categories), and generally uses the same notations and terminology as Mac Lane (MacLane, 1998), except that composition is denoted by ";" and written in the diagrammatic order. The application of functions (functors) to arguments may be written either normally using parentheses, or else in diagrammatic order without parentheses, or, more rarely, by using sub-scripts or super-scripts. The category of sets is denoted as $\mathbb{S}et$, and the category of categories[1] as $\mathbb{C}at$. The opposite of a category $\mathbb{C}$ is denoted by $\mathbb{C}^{\mathrm{op}}$. The class of objects of a category $\mathbb{C}$ is denoted by $|\mathbb{C}|$; also the set of arrows in $\mathbb{C}$ having the object a as source and the object b as target is denoted as $\mathbb{C}(a,b)$. We use $\Rightarrow$ to denote 2-cells in 2-categories. The 'horizontal' composition between 2-cells is also written in diagrammatic order by simple juxtaposition.

[1] We steer clear of any foundational problem related to the "category of all categories"; several solutions can be found in the literature, see, for example (MacLane, 1998).

Indexed categories (Paré and Schumacher, 1978) play an important rôle in this paper, for the purpose of this work they are more adequate than the *fibred categories* (Grothendieck, 1963) formulation of indexation. (Tarlecki et al., 1991) constitutes a good reference for indexed categories and their applications to algebraic specification. An *indexed category* (Tarlecki et al., 1991) is a functor $B\colon I^{\mathrm{op}} \to \mathbb{C}at$; sometimes we denote $B(i)$ as B_i (or B^i) for an index $i \in |I|$ and $B(u)$ as B^u for an index morphism $u \in I$. The following 'flattening' construction providing the canonical fibration associated to an indexed category is known under the name of the *Grothendieck construction* and plays an important rôle in mathematics and in particular in this paper. Given an indexed category $B\colon I^{\mathrm{op}} \to \mathbb{C}at$, let $B^\sharp$ be the *Grothendieck category* having $\langle i, \Sigma\rangle$, with $i \in |I|$ and $\Sigma \in |B_i|$, as objects and $\langle u, \varphi\rangle\colon \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$, with $u \in I(i, i')$ and $\varphi\colon \Sigma \to \Sigma' B^u$, as arrows. The composition of arrows in $B^\sharp$ is defined by $\langle u, \varphi\rangle; \langle u', \varphi'\rangle = \langle u;u', \varphi;(\varphi' B^u)\rangle$.

The following simple lemma will be used later in the paper:

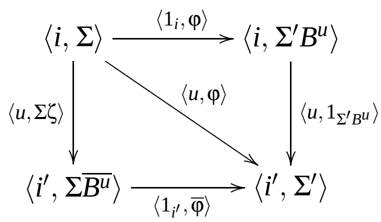
LEMMA 1. *Let $B\colon I^{\mathrm{op}} \to \mathbb{C}at$ be an indexed category. Then each arrow $\langle u, \varphi\rangle\colon \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$ in the Grothendieck category $B^\sharp$ can be canonically factored as*

$$\langle u, \varphi\rangle = \langle 1_i, \varphi\rangle; \langle u, 1_{\Sigma' B^u}\rangle$$

Moreover, if the functor B^u has a left adjoint $\overline{B^u}$ with unit ζ, then $\langle u, \varphi\rangle\colon \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$ can also be factored as

$$\langle u, \varphi\rangle = \langle u, \Sigma\zeta\rangle; \langle 1_{i'}, \overline{\varphi}\rangle$$

where $\overline{\varphi}\colon \Sigma\overline{B^u} \to \Sigma'$ is the free extension of $\varphi\colon \Sigma \to \Sigma' B^u$.



2.1.1. *Grothendieck Construction in 2-categories*

In this section we internalise the Grothendieck construction for indexed categories to any 2-category rather than $\mathbb{C}at$ by using the following basic result:[2]

THEOREM 1. *The Grothendieck category $B^\sharp$ of an indexed category $B\colon I^{\mathrm{op}} \to \mathbb{C}at$ is the vertex of the lax colimit $\mu\colon B \rightsquigarrow B^\sharp$ of B in $\mathbb{C}at$, where*

- *for each index $i \in |I|$, $\mu^i\colon B^i \to B^\sharp$ is the canonical inclusion of categories, and*
- *for each index morphism $u \in I(i, j)$, $\mu^u\colon B^u;\mu^i \Rightarrow \mu^j$ is defined by $\mu^u_b = \langle u, 1_{bB^u}\rangle$ for each object $b \in |B^j|$.*

Lax colimits (see (Borceux, 1994)) constitute the most relaxed concept of colimit in 2-categories, where diagrams are required to commute up to 2-cells only (rather than ordinary strict equality). Notice that since the Grothendieck construction is a lax colimit of an ordinary (1-)functor, this simply means that the lax co-cone μ of Theorem 1 is initial.

COROLLARY 1. *Any 2-functor $B\colon I^* \to \mathbb{C}at$, where I^* is the 2-dimensional dual changing the direction of 2-cells both horizontally and vertically, induces a canonical 2-category structure on the Grothendieck category $B^\sharp$ of the (1)-functor $B\colon I^{\mathrm{op}} \to \mathbb{C}at$.*

[2] We omit the proof of this result since we believe this is mathematical folklore although we are not aware of any clear reference for this result. Also, the proof of this theorem is straightforward.

We now internalise the concept of Grothendieck construction to 2-categories as follows:

DEFINITION 1. Given a (1-)functor $B\colon I^{\mathrm{op}} \to V$, where V is a 2-category, a *Grothendieck construction* for B is a lax colimit $\mu\colon B \rightsquigarrow B^\sharp$. Then $B^\sharp$ is called the *Grothendieck object* associated to B. □

2.2. INSTITUTIONS

Institutions (Goguen and Burstall, 1992) were introduced in the mid eighties as (categorical) abstract model theory for specification and programming; since then the theory of institutions became the modern level of algebraic specification and institutions now constitute the mathematical structure underlying the algebraic specification theory. In this section we briefly review some of the basic concepts on institutions. Besides the seminal paper (Goguen and Burstall, 1992), (Diaconescu et al., 1993) contains many results about institutions with direct application to modularisation in algebraic specification languages.

From a logic perspective, institutions are much more abstract than Tarski's model theory, and also have another basic ingredient, namely signatures and the possibility of translating sentences and models across signature morphisms. A special case of this translation is familiar in first order model theory: if $\Sigma \to \Sigma'$ is an inclusion of first order signatures and M is a Σ'-model, then we can form the *reduct* of M to Σ, denoted $M\restriction_\Sigma$. Similarly, if e is a Σ-sentence, we can always view it as a Σ'-sentence (but there is no standard notation for this). The key axiom, called the *satisfaction condition*, says that *truth is invariant under change of notation*, which is surely a very basic intuition for traditional logic.

DEFINITION 2. An *institution* $\mathfrak{I} = (\mathbb{S}ign, Sen, \mathrm{MOD}, \models)$ consists of

1. a category $\mathbb{S}ign$, whose objects are called *signatures*,
2. a functor $Sen\colon \mathbb{S}ign \to \mathbb{S}et$, giving for each signature a set whose elements are called *sentences* over that signature,
3. a functor $\mathrm{MOD}\colon \mathbb{S}ign^{\mathrm{op}} \to \mathbb{C}at$ giving for each signature Σ a category whose objects are called Σ-*models*, and whose arrows are called Σ-(*model*) *morphisms*, and
4. a relation $\models_\Sigma \subseteq |\mathrm{MOD}(\Sigma)| \times Sen(\Sigma)$ for each $\Sigma \in |\mathbb{S}ign|$, called Σ-*satisfaction*,

such that for each morphism $\varphi\colon \Sigma \to \Sigma'$ in $\mathbb{S}ign$, the *satisfaction condition*

$$m' \models_{\Sigma'} Sen(\varphi)(e) \text{ iff } \mathrm{MOD}(\varphi)(m') \models_\Sigma e$$

holds for each $m' \in |\mathrm{MOD}(\Sigma')|$ and $e \in Sen(\Sigma)$. We may denote the reduct functor $\mathrm{MOD}(\varphi)$ by $\_\restriction_\varphi$ and the sentence translation $Sen(\varphi)$ by $\varphi(\_)$. □

DEFINITION 3. Let $\mathfrak{I} = (\mathbb{S}ign, Sen, \mathrm{MOD}, \models)$ be an institution. For any signature Σ the closure of a set E of Σ-sentences is $E^\bullet = \{e \mid E \models_\Sigma e\}$[3]. (Σ, E) is a *theory* if and only if E is closed, i.e., $E = E^\bullet$.

A *theory morphism* $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ is a signature morphism $\varphi\colon \Sigma \to \Sigma'$ such that $\varphi(E) \subseteq E'$. Let $\mathbb{T}h(\mathfrak{I})$ denote the category of all theories in $\mathfrak{I}$. □

For any institution $\mathfrak{I}$, the model functor MOD extends from the category of its signatures $\mathbb{S}ign$ to the category of its theories $\mathbb{T}h(\mathfrak{I})$, by mapping a theory (Σ, E) to the full subcategory $\mathrm{MOD}(\Sigma, E)$ of $\mathrm{MOD}(\Sigma)$ formed by the Σ-models which satisfy E.

[3] $E \models_\Sigma e$ means that $M \models_\Sigma e$ for any Σ-model M that satisfies all sentences in E.

DEFINITION 4. A theory morphism $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ is *liberal* if and only if the reduct functor $\_\!\restriction_\varphi\colon \mathrm{MOD}(\Sigma', E') \to \mathrm{MOD}(\Sigma, E)$ has a left-adjoint $(\_)^\varphi$.

The institution $\mathfrak{I}$ is *liberal* if and only if each theory morphism is liberal. □

General results (Tarlecki, 1986) show that liberality is equivalent to the power of Horn axiomatisability.

DEFINITION 5. An institution $\mathfrak{I} = (\mathbb{S}ign, Sen, \mathrm{MOD}, \models)$ is *exact* if and only if the model functor $\mathrm{MOD}\colon \mathbb{S}ign^{\mathrm{op}} \to \mathbb{C}at$ preserves finite limits. $\mathfrak{I}$ is *semi-exact* if and only if MOD preserves only pullbacks. □

Exactness properties for institutions formalise the possibility of amalgamating models of different signatures when they are consistent on some kind of 'intersection' of the signatures (formalised as a pullback). In practice, the weak[4] version of exactness properties may actually suffice (see (Diaconescu, 1998; Tarlecki, 2000)).

2.2.1. *Institution morphisms*

DEFINITION 6. Let $\mathfrak{I}$ and $\mathfrak{I}'$ be institutions. Then an *institution morphism* $\mathfrak{I}' \to \mathfrak{I}$ consists of

1. a functor $\Phi\colon \mathbb{S}ign' \to \mathbb{S}ign$,
2. a natural transformation $\alpha\colon \Phi; Sen \Rightarrow Sen'$, and
3. a natural transformation $\beta\colon \mathrm{MOD}' \Rightarrow \Phi^{\mathrm{op}}; \mathrm{MOD}$

such that the following *satisfaction condition* holds

$$m' \models'_{\Sigma'} \alpha_{\Sigma'}(e) \ \text{ iff } \ \beta_{\Sigma'}(m') \models_{\Sigma'\Phi} e$$

for any Σ'-model m' from $\mathfrak{I}'$ and any $\Sigma'\Phi$-sentence e from $\mathfrak{I}$.

An institution *modification* between institution morphisms $(\Phi, \alpha, \beta) \Rightarrow (\Phi', \alpha', \beta')$ consists of

1. a natural transformation $\tau\colon \Phi \Rightarrow \Phi'$,
2. a modification $\omega\colon \beta \Rightarrow \beta'; \tau\mathrm{MOD}$, i.e., for each $\Sigma' \in |\mathbb{S}ign'|$, a natural transformation $\omega_{\Sigma'}\colon \beta_{\Sigma'} \Rightarrow \beta'_{\Sigma'}; \mathrm{MOD}(\tau_{\Sigma'})$.

□

By defining the canonical compositions (both vertical and horizontal) for institution morphisms and modifications, we can define a 2-category $\mathbb{I}ns$ which has institutions as objects (0-cells), institution morphisms as 1-cells, and their modifications as 2-cells.

In the literature there are several other concepts of institution homomorphism (such as the so-called "institution representations"), each of them being adequate to some specific class of problems; a survey on this topic can be found in (Tarlecki, 1996). The definition presented above and originally given by (Goguen and Burstall, 1992) intuitively expresses that a "richer" institution is built over a "poorer" one, and is the most relevant for the applications of this work. This definition is also a structure preserving one (as will be seen in Section 3).

The following properties of institution morphisms play an important rôle in this paper.

DEFINITION 7. An institution morphism $(\Phi, \alpha, \beta)\colon \mathfrak{I}' \to \mathfrak{I}$ is

- an *equivalence* iff Φ is an equivalence of categories,
- an *embedding* iff Φ admits a left-adjoint $\overline{\Phi}$ (with unit ζ); an institution embedding is denoted as $(\Phi, \overline{\Phi}, \zeta, \alpha, \beta)\colon \mathfrak{I}' \to \mathfrak{I}$, and is

[4] In the sense of 'weak universal properties' of (MacLane, 1998) requiring only *existence* without uniqueness for the corresponding universal arrows.

– *liberal* iff $\beta_{\Sigma'}$ has a left-adjoint $\overline{\beta}_{\Sigma'}$ for each $\Sigma' \in |\mathbb{S}ign'|$.

An institution embedding $(\Phi, \overline{\Phi}, \zeta, \alpha, \beta)\colon \mathfrak{I}' \to \mathfrak{I}$ is *exact* if and only if the square below is a pullback

$$\begin{array}{ccc}
\mathrm{MOD}(\Sigma) & \xleftarrow{\mathrm{MOD}(\varphi)} & \mathrm{MOD}(\Sigma_1) \\
{\scriptstyle \mathrm{MOD}(\Sigma\zeta)}\uparrow & & \uparrow{\scriptstyle \mathrm{MOD}(\Sigma_1\zeta)} \\
\mathrm{MOD}(\Sigma\overline{\Phi}\Phi) & & \mathrm{MOD}(\Sigma_1\overline{\Phi}\Phi) \\
{\scriptstyle \beta_{\Sigma\overline{\Phi}}}\uparrow & & \uparrow{\scriptstyle \beta_{\Sigma_1\overline{\Phi}}} \\
\mathrm{MOD}'(\Sigma\overline{\Phi}) & \xleftarrow[\mathrm{MOD}'(\varphi\overline{\Phi})]{} & \mathrm{MOD}'(\Sigma_1\overline{\Phi})
\end{array}$$

where $\varphi\colon \Sigma \to \Sigma_1$ is any signature morphism in $\mathfrak{I}$. □

Our notion of institution equivalence is a natural generalisation of the notion of categorical equivalence. The idea of institution embedding (although not formulated directly) is as old as the seminal work on institutions (Goguen and Burstall, 1992). Notice that the terminology ‘institution embedding’ is used also by (Meseguer, 1998) but in a completely different sense. Besides (Diaconescu, 1998), several stronger variants of liberal institution morphisms have been independently introduced in the literature, such that the *categorical retractive simulations* of (Kreowski and Mossakowski, 1995) and the *extension maps* of (Meseguer, 1998). Exact institution embeddings are a novel concept which expresses the primitive possibility of amalgamation of consistent models across an institution embedding. Similar notions of exactness based on β-naturality diagrams being pullbacks have been introduced in the literature, such as the *additive institution morphisms* of (Diaconescu, 1998; Diaconescu and Stefaneas, 1998) and the *institutions representations with amalgamation* of (Tarlecki, 2000).

3. Grothendieck Institutions

The following definition generalises the concept of indexed category to institutions.

DEFINITION 8. An *indexed institution* $\mathcal{J}$ is a functor $\mathcal{J}\colon I^{\mathrm{op}} \to \mathbb{I}ns$. □

The following theorem generalises the Grothendieck construction from categories to institutions:

THEOREM 2. *The 2-category of institutions $\mathbb{I}ns$ admits a Grothendieck construction for each indexed institution $\mathcal{J}\colon I^{\mathrm{op}} \to \mathbb{I}ns$.*

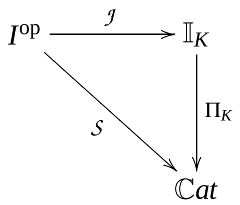
Proof. [5]

We start with the following lemma:

LEMMA 2. *Let K be any 2-category and $\mathbb{I}_K$ be the Grothendieck 2-category for the 2-functor $\mathbb{C}at[(-)^{\mathrm{op}}, K]\colon \mathbb{C}at^* \to \mathbb{C}at$. Then the fibration $\Pi_K\colon \mathbb{I}_K \to \mathbb{C}at$ creates Grothendieck constructions for each functor $\mathcal{J}\colon I^{\mathrm{op}} \to \mathbb{I}_K$.*

Proof. We have to prove that for each functor $\mathcal{J}\colon I^{\mathrm{op}} \to \mathbb{I}_K$, there exists a Grothendieck construction $\mu\colon \mathcal{J} \rightsquigarrow \mathcal{J}^\sharp$ in $\mathbb{I}_K$ such that $\Pi_K(\mu) = \mu_K$, where $\mu_K\colon \mathcal{S} \rightsquigarrow \mathcal{S}^\sharp$ is the Grothendieck construction in $\mathbb{C}at$ for $\mathcal{S} = \mathcal{J};\Pi_K$.

[5] For a better understanding of the structure of Grothendieck institutions we go here for a rather direct proof of this result. Alternatively one may use the general theorem of existence of weighted colimits in enriched categories (Borceux, 1994) instantiated to the case of lax colimits.



Since $\mathbb{I}_K$ is a Grothendieck (2-)category, as notational convention, let us assume that $\mathcal{J}^u = \langle \mathcal{S}^u, \mathfrak{I}^u \rangle$ for each $u \in I$ (either index or index morphism).

Let $\mathcal{J}^\sharp = \langle \mathcal{S}^\sharp, \mathfrak{I}^\sharp \rangle$, where $\mathfrak{I}^\sharp \colon (\mathcal{S}^\sharp)^{\mathrm{op}} \to K$ is the unique functor (by the universal property of the Grothendieck construction for $\mathcal{S}$) such that

– $\mu^i_K; (\mathfrak{I}^\sharp)^{\mathrm{op}} = (\mathfrak{I}^i)^{\mathrm{op}}$ for each object $i \in |I|$, and

– $\mu^u_K (\mathfrak{I}^\sharp)^{\mathrm{op}} = (\mathfrak{I}^u)^{\mathrm{op}}$ for each arrow $u \in I$.

We then define μ by

$$\mu^u = \langle \mu^u_K, 1_{\mathfrak{I}^u} \rangle$$

for each $u \in I$ (either index or index morphism) and we have to prove that μ is initial. This is enough since lax colimits of (ordinary 1-)functors are simply initial lax co-cones. Consider another lax co-cone $\langle \nu, \rho \rangle \colon \mathcal{J} \rightsquigarrow \langle \mathcal{S}', \mathfrak{I}' \rangle$. We prove that there exists a unique $\langle \nu', \rho' \rangle \colon \langle \mathcal{S}^\sharp, \mathfrak{I}^\sharp \rangle \to \langle \mathcal{S}', \mathfrak{I}' \rangle$ such that

– $\langle \mu^i_K, 1_{\mathfrak{I}^i} \rangle ; \langle \nu', \rho' \rangle = \langle \nu^i, \rho^i \rangle$ for each index $i \in |I|$, and

– $\langle \mu^u_K, 1_{\mathfrak{I}^u} \rangle \langle \nu', \rho' \rangle = \langle \nu^u, \rho^u \rangle$ for each index morphism $u \in I$.

By projecting the first condition on the first component, we have that $\nu' \colon \mathcal{S}^\sharp \to \mathcal{S}'$ is the unique functor such that $\mu^i_K; \nu' = \nu^i$ for each index $i \in |I|$ and $\mu^u_K \nu' = \nu^u$ for each index morphism $u \in I$.

The first condition on the second component means $(\mu^i_K)^{\mathrm{op}} \rho' = \rho^i$ for each $i \in |I|$, which determines the natural transformation ρ' by $\rho'_{\langle i, \Sigma \rangle} = \rho^i_\Sigma$ for each $i \in |I|$ and $\Sigma \in |\mathcal{S}^i|$. The checking of the second condition follows now by routine calculations.

This concludes the proof of this lemma.

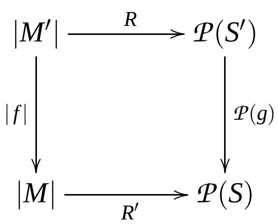
The theorem now follows by noticing that $\mathbb{I}ns = \mathbb{I}_{\mathbb{R}oom}$, where $\mathbb{R}oom$ is the 2-category which has objects triples (M, S, R) such that

– M is a category,

– S is a set (regarded as discrete small category), and

– R is a function $|M| \to \mathcal{P}(S)$, where $\mathcal{P}$ is the contravariant power-set functor,

and has pairs of functors $\langle M' \xrightarrow{f} M, S \xrightarrow{g} S' \rangle$ as 1-cells $(M', S', R') \to (M, S, R)$ such that the following diagram



commutes, and has natural transformations $f \Rightarrow f'$ as 2-cells between $\langle f, g \rangle \Rightarrow \langle f', g' \rangle$.

Notice that the generality level of Lemma 2 permits variants of Theorem 2 for concepts of institutions enriched with additional structure, such as proof-theoretic, operational, etc. This can be easily achieved by replacing $\mathbb{R}oom$ with an appropriate 2-category.

The explicit structure of the Grothendieck institution of an indexed institution is given by the following:

REMARK 1. The Grothendieck institution $\mathcal{J}^\sharp$ of an indexed institution $\mathcal{J}\colon I^{\mathrm{op}} \to \mathbb{I}ns$ has

1. the Grothendieck category $\mathbb{S}ign^\sharp$ as its category of signatures, where $\mathbb{S}ign\colon I^{\mathrm{op}} \to \mathbb{C}at$ is the *indexed* category of signatures of the indexed institution $\mathfrak{I}$,
2. $\mathrm{MOD}^\sharp\colon (\mathbb{S}ign^\sharp)^{\mathrm{op}} \to \mathbb{C}at$ as its model functor, where
   - $\mathrm{MOD}^\sharp(\langle i, \Sigma\rangle) = \mathrm{MOD}^i(\Sigma)$ for each index $i \in |I|$ and signature $\Sigma \in |\mathbb{S}ign^i|$, and
   - $\mathrm{MOD}^\sharp(\langle u, \varphi\rangle) = \beta^u_{\Sigma'}; \mathrm{MOD}^i(\varphi)$ for each $\langle u, \varphi\rangle\colon \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$,
3. $Sen^\sharp\colon \mathbb{S}ign^\sharp \to \mathbb{S}et$ as its sentence functor, where
   - $Sen^\sharp(\langle i, \Sigma\rangle) = Sen^i(\Sigma)$ for each index $i \in |I|$ and signature $\Sigma \in |\mathbb{S}ign^i|$, and
   - $Sen^\sharp(\langle u, \varphi\rangle) = Sen^i(\varphi); \alpha^u_{\Sigma'}$ for each $\langle u, \varphi\rangle\colon \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$,
4. $m \models^\sharp_{\langle i,\Sigma\rangle} e$ iff $m \models^i_\Sigma e$ for each index $i \in |I|$, signature $\Sigma \in |\mathbb{S}ign^i|$, model $m \in |\mathrm{MOD}^\sharp(\langle i, \Sigma\rangle)|$, and sentence $e \in Sen^\sharp(\langle i, \Sigma\rangle)$.

where $\mathcal{J}^i = (\mathbb{S}ign^i, \mathrm{MOD}^i, Sen^i, \models^i)$ for each index $i \in |I|$ and $\mathcal{J}^u = (\Phi^u, \alpha^u, \beta^u)$ for $u \in I$ index morphism. □

COROLLARY 2. *The concept of extra theory morphism (*Diaconescu, 1998*) across an institution morphism $\mathfrak{I}' \to \mathfrak{I}$ (with all its subsequent concepts) is recuperated as an ordinary theory morphism in the Grothendieck institution of the indexed institution given by the morphism $\mathfrak{I}' \to \mathfrak{I}$ (i.e., which has $\bullet \to \bullet$ as its index category).*

3.1. FIBRED INSTITUTIONS

For the readers preferring fibred categories to indexed categories, we generalise fibred categories to *fibred institutions*. We show that *split fibred institutions* are essentially the same as the (previously introduced) Grothendieck institutions. Readers with no background in fibred categories may skip this section because the rest of the paper does not use any of the developments of this section and stays within the framework of indexed and Grothendieck institutions. For this reason we also keep the developments of this section very brief.

DEFINITION 9. Given a category I, a *fibred institution over the base I* is a tuple $\mathfrak{I} = (\mathbb{S}ign, I, \Pi, \mathrm{MOD}, Sen, \models)$ such that

- $\Pi\colon \mathbb{S}ign \to I$ is a fibred category, and
- $(\mathbb{S}ign, \mathrm{MOD}, Sen, \models)$ is an institution.

$\mathfrak{I}$ is *split* when the fibration Π is split.

A *cartesian institution morphism* is an institution morphism between fibred institutions for which the signature mapping functor is cartesian functor between the corresponding fibred categories of signatures. □

EXAMPLE 1. By Remark 1 any Grothendieck institution is a split fibred institution. □

Now we try to define an opposite mapping, from (split) fibred institutions to indexed institutions.

DEFINITION 10. Given a fibred institution $\mathfrak{I} = (\mathbb{S}ign, I, \Pi, \mathrm{MOD}, Sen, \models)$, for each object $i \in |I|$, the *fibre of* $\mathfrak{I}$ *at* i is the institution $\mathfrak{I}^i = (\mathbb{S}ign^i, \mathrm{MOD}^i, Sen^i, \models^i)$ where

- $\mathbb{S}ign^i$ is the fibre of Π at i, and
- MOD^i, Sen^i, and $\models^i$ are the restrictions of MOD, Sen, and respectively $\models$ to $\mathbb{S}ign^i$.

□

PROPOSITION 1. *Given a fibred institution* $\mathfrak{I} = (\mathbb{S}ign, I, \Pi, \mathrm{MOD}, Sen, \models)$, *for each arrow* $u \in I(i,j)$, *any 'inverse image functor'* $\Phi^u\colon \mathbb{S}ign^j \to \mathbb{S}ign^i$ *determines a canonical institution morphism* $(\Phi^u, \alpha^u, \beta^u)\colon \mathfrak{I}^j \to \mathfrak{I}^i$ *between the fibres of* $\mathfrak{I}$, *where* $\alpha^u_{\Sigma'} = Sen(\varphi^{\Phi}_{\Sigma'})$ *and* $\beta^u_{\Sigma'} = \mathrm{MOD}(\varphi^{\Phi}_{\Sigma'})$ *for each signature* Σ' *in the fibre* $\mathbb{S}ign^j$ *at* j, *and* $\varphi^{\Phi}_{\Sigma'}\colon \Sigma'\Phi^u \to \Sigma'$ *being the distinguished cartesian morphism corresponding to* Φ^u.

Proof. The naturality of α^u and β^u follow directly from the way the family of distinguished cartesian morphisms $\{\varphi^{\Phi}_{\Sigma'}\}_{\Sigma'}$ determine the functor Φ^u, and by applying the sentence functor and the model functor, respectively, to the corresponding commutative diagrams.

Finally, the satisfaction condition for the institution morphism $(\Phi^u, \alpha^u, \beta^u)$ follows from the satisfaction condition of the fibred institution $\mathfrak{I}$ applied for the distinguished cartesian morphisms.

COROLLARY 3. *Consider a category* I. *There exists a natural isomorphism between the category of split fibred institutions over* I *(with cartesian institution morphisms as arrows) and the category of institutions indexed by* I *(with natural transformation between the indexing functors as arrows).*

Now we can extend Bénabou's result (Bénabou, 1985) to fibred institutions:

COROLLARY 4. *Each fibred institution is equivalent to a Grothendieck institution.*

4. Globalisation of Institutional Properties

This section is devoted to the study of the most important institutional properties (as seen from the semantics of specification languages; see (Diaconescu et al., 1993)) for Grothendieck institutions. These include theory colimits, liberality (i.e., free constructions), exactness (i.e., model amalgamation), and inclusion systems for institutions. In all cases we follow the same pattern of 'globalisation' of the properties by lifting them from the 'local' level of the indexed institution to the 'global' level of the Grothendieck institution. All developments of this section can be immediately translated into the language of fibred categories/institutions. However, the framework of indexed institutions seems to be the most appropriate for applications and for the presentations and development of the results.

Most of the developments of this section rely on a stronger version of indexed institution for which the institution morphisms are embeddings. This is a natural technical condition which almost always occurs in practical applications.

DEFINITION 11. An *embedding-indexed institution* is an indexed institution $\mathcal{J}\colon I^{\mathrm{op}} \to \mathbb{I}ns$ for which all institution morphisms $\mathcal{J}^u$ are embeddings $(\Phi^u, \overline{\Phi^u}, \zeta^u, \alpha^u, \beta^u)$ for all index morphisms $u \in I$.

A embedding-indexed institution is *coherent* if and only if

$\overline{\Phi^u}; \overline{\Phi^{u'}} = \overline{\Phi^{u;u'}}$ (i.e., the indexed category of signatures is 'globally' reversible)

and

$$\zeta^u; \overline{\Phi^u}\zeta^{u'}\Phi^u = \zeta^{u;u'}$$

for each $u \in I(i,j)$ and $u' \in I(j,k)$. □

4.1. Theory Colimits in Grothendieck Institutions

DEFINITION 12. An indexed category $B\colon I^{\mathrm{op}} \to \mathbb{C}at$ is *locally J-cocomplete* for a small category J if and only if the category B^i is J-cocomplete for each index $i \in |I|$. □

The ‘sufficient’ part of the following fundamental result was essentially obtained for the first time in (Diaconescu, 1998) in the context of ‘extra theory morphisms’.

THEOREM 3. *Let $\mathcal{J}\colon I^{\mathrm{op}} \to \mathbb{I}ns$ be an embedding-indexed institution such that I is J-cocomplete for a small category J. Then the category of theories $\mathbb{T}h(\mathcal{J}^\sharp)$ of the Grothendieck institution $\mathcal{J}^\sharp$ has J-colimits if and only if the indexed category of signatures $\mathbb{S}ign$ of $\mathcal{J}$ is locally J-cocomplete.*

Proof. For the ‘necessary’ part of this theorem, it is sufficient to notice that for each index $i \in I$, the canonical inclusion functor $\mathbb{T}h(\mathcal{J}^i) \hookrightarrow \mathbb{T}h(\mathcal{J}^\sharp)$ reflects colimits, hence $\mathbb{T}h(\mathcal{J}^i)$ has J-colimits if $\mathbb{T}h(\mathcal{J}^\sharp)$ has J-colimits. This implies that $\mathbb{S}ign^i$ has J-colimits for each index $i \in I$.

For the ‘sufficient’ part of the theorem, by the fundamental result that in any institution the forgetful functor from theories to signatures creates colimits (see (Goguen and Burstall, 1992)), we have only to show that the category of signatures of the Grothendieck institution $\mathcal{J}^\sharp$ has J-colimits. By Remark 1, the category of signatures of $\mathcal{J}^\sharp$ is the Grothendieck category of signatures $\mathbb{S}ign^\sharp$. The conclusion of the theorem now follows from the general result on existence of colimits in Grothendieck categories (see (Tarlecki et al., 1991)).

This theorem shows very clearly the conceptual power of Grothendieck institutions, since in this case they enable a very compact proof by invoking important results from indexed category theory. This situation contrasts to the rather complex proof given in (Diaconescu, 1998) for the existence of colimits for extra theory morphisms.

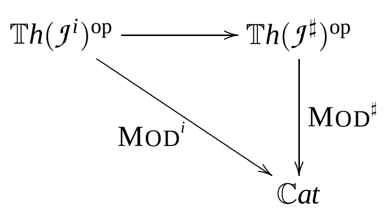
4.2. Liberality in Grothendieck Institutions

DEFINITION 13. An indexed institution $\mathcal{J}\colon I^{\mathrm{op}} \to \mathbb{I}ns$ is *locally liberal* if and only if the institution $\mathcal{J}^i$ is liberal for each index $i \in I$. □

The following result represents the global counterpart of a similar result of (Diaconescu, 1998) where we studied liberality at the level theory morphisms only.

THEOREM 4. *The Grothendieck institution $\mathcal{J}^\sharp$ of an indexed institution $\mathcal{J}\colon I^{\mathrm{op}} \to \mathbb{I}ns$ is liberal if and only if $\mathcal{J}$ is liberal and each institution morphism $\mathcal{J}^u$ is liberal for each index morphism $u \in I$.*

Proof. The ‘necessary’ part of the theorem follows by noticing that local liberality of the indexed institution is contained by the liberality of the Grothendieck institution because each model functor $\mathrm{M}\textsc{od}^i$ is a restriction of the model functor $\mathrm{M}\textsc{od}^\sharp$ of the Grothendieck institution $\mathcal{J}^\sharp$:



and by noticing that for each index morphism $u \in I(i,i')$, the liberality of the institution morphism $\mathcal{J}^u = (\Phi^u, \alpha^u, \beta^u)$ is the same as the liberality of the (extra) signature morphism $\langle u, 1_{\Sigma'\Phi^u}\rangle\colon \langle i, \Sigma'\Phi^u\rangle \to \langle i', \Sigma'\rangle$.

For the ‘sufficient’ part of the theorem, we consider an (extra) theory morphism $\langle u, \varphi\rangle\colon \langle i, (\Sigma, E)\rangle \to \langle i', (\Sigma', E')\rangle$ and a (Σ, E)-model M. The free expansion of M along $\langle u, \varphi\rangle$ is the model $\overline{\beta_{\Sigma'}}(M^\varphi)/E'$, where M^φ is the free expansion of M along the (intra) signature morphism $\varphi\colon \Sigma \to \Sigma'\Phi^u$ (by the liberality of $\mathcal{J}^i$), and $\_/E'$ is the left adjoint to the forgetful inclusion $\mathrm{M}\textsc{od}^{i'}(\Sigma', E') \hookrightarrow \mathrm{M}\textsc{od}^{i'}(\Sigma')$ (by the liberality of $\mathcal{J}^{i'}$). Finally, the universal property of $\overline{\beta_{\Sigma'}}(M^\varphi)/E'$

follows as a composition of the three universal properties corresponding to the three adjunctions involved:

$$
\begin{array}{ccc}
M & \longrightarrow & M^{\varphi}\restriction_{\varphi} \\
& \searrow^{h} & \downarrow h^{\varphi}\restriction_{\varphi} \\
& & \beta_{\Sigma'}(N)\restriction_{\varphi}
\end{array}
\qquad
\begin{array}{ccc}
M^{\varphi} & \longrightarrow & \beta_{\Sigma'}(\overline{\beta_{\Sigma'}}(M^{\varphi})) \\
& \searrow^{h^{\varphi}} & \downarrow \beta_{\Sigma'}(\overline{h^{\varphi}}) \\
& & \beta_{\Sigma'}(N)
\end{array}
\qquad
\begin{array}{ccc}
\overline{\beta_{\Sigma'}}(M^{\varphi}) & \longrightarrow & \overline{\beta_{\Sigma'}}(M^{\varphi})/E' \\
& \searrow^{\overline{h^{\varphi}}} & \downarrow \overline{h^{\varphi}}/E' \\
& & N \models E'
\end{array}
$$

This liberality result is also stronger than its counterpart from (Diaconescu, 1998) because it gives an ‘if and only if’ characterization of liberality in Grothendieck institutions.

4.3. Exactness in Grothendieck Institutions

From all the properties of Grothendieck institutions, exactness seems to be the most complex to study. In (Diaconescu, 1998) we conjectured an ‘if and only if’ characterization of exactness for extra theory morphisms, in this section we solve this problem. Our approach is to decompose the exactness property into a set of atomic orthogonal necessary and sufficient conditions.

DEFINITION 14. An indexed institution $\mathcal{J}\colon I^{\mathrm{op}} \to \mathbb{I}ns$ is *locally (semi-)exact* if and only if the institution $\mathcal{J}^i$ is (semi-)exact for each index $i \in I$. □

PROPOSITION 2. *If the Grothendieck institution of an indexed institution is semi-exact, then the indexed institution is locally semi-exact.*

Proof. By Remark 1, for each index i, the model functor MOD^i is the restriction $\mathrm{MOD}^{\sharp}(\langle i, -\rangle)$ of the model functor of the Grothendieck institution to the sub-category $\mathbb{S}ign^i$ of the Grothendieck category of signatures $\mathbb{S}ign^{\sharp}$ (i.e. the category of signatures of the Grothendieck institution).

$$
\begin{array}{ccc}
(\mathbb{S}ign^i)^{\mathrm{op}} & \longrightarrow & (\mathbb{S}ign^{\sharp})^{\mathrm{op}} \\
& \searrow^{\mathrm{MOD}^i} & \downarrow \mathrm{MOD}^{\sharp} \\
& & \mathbb{C}at
\end{array}
$$

Because the canonical injection $\mathbb{S}ign^i \to \mathbb{S}ign^{\sharp}$ preserves co-limits (as a simple general property of the Grothendieck constructions), we have that MOD^i preserves whatever limits are preserved by $\mathrm{MOD}^{\sharp}$, hence MOD^i preserves pullbacks.

PROPOSITION 3. *If the Grothendieck institution of an indexed institution is semi-exact, then each institution embedding of the indexed institution is exact.*

Proof. Consider an institution embedding $(\Phi^u, \overline{\Phi^u}, \zeta^u, \alpha^u, \beta^u)\colon \mathcal{J}^{i'} \to \mathcal{J}^i$, and an arbitrary signature morphism $\varphi\colon \Sigma \to \Sigma_1$ in $\mathcal{J}^i$.

Notice that the following square

$$
\begin{array}{ccc}
\langle i, \Sigma\rangle & \xrightarrow{\langle 1, \varphi\rangle} & \langle i, \Sigma_1\rangle \\
\downarrow{\scriptstyle \langle u, \Sigma\zeta\rangle} & & \downarrow{\scriptstyle \langle u, \Sigma_1\zeta\rangle} \\
\langle i', \Sigma\overline{\Phi}\rangle & \xrightarrow[\langle 1, \varphi\overline{\Phi}\rangle]{} & \langle i', \Sigma_1\overline{\Phi}\rangle
\end{array}
$$

is a pushout. Because the Grothendieck institution is semi-exact, this pushout is mapped by the (Grothendieck) model functor to a pullback square. All we still have to do is to notice that this pullback square gives the exactness of the institution embedding $(\Phi^u, \overline{\Phi^u}, \zeta^u, \alpha^u, \beta^u)$.

DEFINITION 15. A coherent embedding-indexed institution $\mathcal{J} : I^{\mathrm{op}} \to \mathbb{I}ns$ is *semi-exact* if and only if for each pushout

$$\begin{array}{ccc} i & \xrightarrow{u1} & j1 \\ {\scriptstyle u2}\downarrow & & \downarrow{\scriptstyle v1} \\ j2 & \xrightarrow[v2]{} & k \end{array}$$

in I, the following square

$$\begin{array}{ccccc}
\mathrm{MOD}^i(\Sigma) & \xleftarrow{\mathrm{MOD}^i(\Sigma\zeta^{u1})} & \mathrm{MOD}^i(\Sigma\overline{\Phi^{u1}}\Phi^{u1}) & \xleftarrow{\beta^{u1}_{\Sigma\overline{\Phi^{u1}}}} & \mathrm{MOD}^{j1}(\Sigma\overline{\Phi^{u1}}) \\
{\scriptstyle \mathrm{MOD}^i(\Sigma\zeta^{u2})}\uparrow & & & & \uparrow{\scriptstyle \mathrm{MOD}^{j1}(\Sigma\overline{\Phi^{u1}}\zeta^{v1})} \\
\mathrm{MOD}^i(\Sigma\overline{\Phi^{u1}}\Phi^{u1}) & & & & \mathrm{MOD}^{j1}(\Sigma\overline{\Phi^{u1}\Phi^{v1}}\Phi^{v1}) \\
{\scriptstyle \beta^{u2}_{\Sigma\overline{\Phi^{u2}}}}\uparrow & & & & \uparrow{\scriptstyle \beta^{v1}_{\Sigma\overline{\Phi^{u1}\Phi^{v1}}}} \\
\mathrm{MOD}^{j2}(\Sigma\overline{\Phi^{u2}}) & \xleftarrow[\mathrm{MOD}^{j2}(\Sigma\overline{\Phi^{u2}}\zeta^{v2})]{} & \mathrm{MOD}^{j2}(\Sigma\overline{\Phi^{u2}\Phi^{v2}}\Phi^{v2}) & \xleftarrow[\beta^{v2}_{\Sigma\overline{\Phi^{u2}\Phi^{v2}}}]{} & \mathrm{MOD}^{k}(\Sigma\overline{\Phi^{ui}\Phi^{vi}})
\end{array}$$

is a pullback for each signature Σ in $\mathfrak{I}^i$. □

PROPOSITION 4. *Let $\mathcal{J}$ be a coherent embedding-indexed institution. If the Grothendieck institution $\mathcal{J}^\sharp$ is semi-exact, then the indexed institution $\mathcal{J}$ is also semi-exact.*

Proof. Consider a pushout square in I as in Definition 15. Notice (by the colimit construction in Grothendieck categories cf. (Tarlecki et al., 1991)) that the following square

$$\begin{array}{ccc}
\langle i, \Sigma\rangle & \xrightarrow{\langle u1, \Sigma\zeta^{u1}\rangle} & \langle j1, \Sigma\overline{\Phi^{u1}}\rangle \\
{\scriptstyle \langle u2, \Sigma\zeta^{u2}\rangle}\downarrow & & \downarrow{\scriptstyle \langle v1, \Sigma\overline{\Phi^{u1}}\zeta^{v1}\rangle} \\
\langle j2, \Sigma\overline{\Phi^{u2}}\rangle & \xrightarrow[\langle v2, \Sigma\overline{\Phi^{u2}}\zeta^{v2}\rangle]{} & \langle k, \Sigma\overline{\Phi^{ui}\Phi^{vi}}\rangle
\end{array}$$

is a pushout in the category of signatures $\mathbb{S}ign^\sharp$ of the Grothendieck institution. Because the Grothendieck institution is semi-exact, the Grothendieck model functor $\mathrm{MOD}^\sharp$ maps this pushout square to the pullback square of Definition 15.

THEOREM 5. *Let $\mathcal{J}$ be a coherent embedding-indexed institution. The Grothendieck institution $\mathcal{J}^\sharp$ is semi-exact if and only if*

– *the indexed institution $\mathcal{J}$ is locally semi-exact,*

– *the indexed institution $\mathcal{J}$ is semi-exact, and*

– *all institution embeddings are exact.*

Proof. The ‘necessary’ part of this theorem holds by Proposition 2, Proposition 4, and Proposition 3.

For the ‘sufficient’ part, we consider an arbitrary pushout of signatures in the Grothendieck institution

$$\begin{array}{ccc}
\langle \mathbb{S}ign^0, \Sigma_0\rangle & \xrightarrow{\langle \Phi^{u1}, \varphi_1\rangle} & \langle \mathbb{S}ign^1, \Sigma_1\rangle \\
{\scriptstyle \langle \Phi^{u2},\varphi_2\rangle}\Big\downarrow & & \Big\downarrow{\scriptstyle\langle \Phi^1,\theta_1\rangle}\\
\langle \mathbb{S}ign^2,\Sigma_2\rangle & \xrightarrow[\langle\Phi^2,\theta_2\rangle]{} & \langle \mathbb{S}ign,\Sigma\rangle
\end{array}$$

where

$$\begin{array}{ccc}
\mathfrak{I}^0 & \xleftarrow{(\Phi^{u1},\overline{\Phi^{u1}},\zeta^{u1},\alpha^{u1},\beta^{u1})} & \mathfrak{I}^1 \\
{\scriptstyle (\Phi^{u2},\overline{\Phi^{u2}},\zeta^{u2},\alpha^{u2},\beta^{u2})}\Big\uparrow & & \Big\uparrow{\scriptstyle (\Phi^1,\overline{\Phi^1},\zeta^1,\alpha^1,\beta^1)}\\
\mathfrak{I}^2 & \xleftarrow[(\Phi^2,\overline{\Phi^2},\zeta^2,\alpha^2,\beta^2)]{} & \mathfrak{I}
\end{array}$$

is the underlying square of institution embeddings.[6]

By factoring each of the extra signature morphisms accordingly to the second part of Lemma 1, and by applying the pushout construction in Grothendieck categories (cf. (Tarlecki et al., 1991)), due to the coherence property of the indexed institution, the pushout square of extra signature morphisms can be expressed as the following composition of four pushout squares:

$$\begin{array}{ccccc}
\langle \mathbb{S}ign^0, \Sigma_0\rangle & \xrightarrow{\langle \Phi^{u1},\Sigma_0\zeta^{u1}\rangle} & \langle \mathbb{S}ign^1, \Sigma_0\overline{\Phi^{u1}}\rangle & \xrightarrow{\langle 1,\overline{\varphi_1}\rangle} & \langle \mathbb{S}ign^1,\Sigma_1\rangle \\
{\scriptstyle\langle \Phi^{u2},\Sigma_0\zeta^{u2}\rangle}\Big\downarrow & & \Big\downarrow{\scriptstyle\langle\Phi^1,\Sigma_0\overline{\Phi^{u1}}\zeta^1\rangle} & & \Big\downarrow{\scriptstyle\langle\Phi^1,\Sigma_1\zeta^1\rangle}\\
\langle \mathbb{S}ign^2,\Sigma_0\overline{\Phi^{u2}}\rangle & \xrightarrow[\langle\Phi^2,\Sigma_0\overline{\Phi^{u2}}\zeta^2\rangle]{} & \langle \mathbb{S}ign,\Sigma_0\overline{\Phi^{ui}\Phi^i}\rangle & \xrightarrow[\langle 1,\overline{\varphi_1\Phi^1}\rangle]{} & \langle \mathbb{S}ign,\Sigma_1\overline{\Phi^1}\rangle \\
{\scriptstyle\langle 1,\overline{\varphi_2}\rangle}\Big\downarrow & & \Big\downarrow{\scriptstyle\langle 1,\overline{\varphi_2\Phi^2}\rangle} & & \Big\downarrow{\scriptstyle\langle 1,\overline{\theta_1}\rangle}\\
\langle \mathbb{S}ign^2,\Sigma_2\rangle & \xrightarrow[\langle\Phi^2,\Sigma_2\zeta^2\rangle]{} & \langle \mathbb{S}ign,\Sigma_2\overline{\Phi^2}\rangle & \xrightarrow[\langle 1,\overline{\theta_2}\rangle]{} & \langle \mathbb{S}ign,\Sigma\rangle
\end{array}$$

The Grothendieck model functor

- maps the up-left pushout square to a pullback square because the indexed institution is semi-exact,
- maps the down-right pushout square to a pullback square because the indexed institution is locally semi-exact, and
- maps the up-right and down-left pushout squares to pullback squares because the institution embeddings $(\Phi^1,\overline{\Phi^1},\zeta^1,\alpha^1,\beta^1)$ and $(\Phi^2,\overline{\Phi^2},\zeta^2,\alpha^2,\beta^2)$ are exact.

Therefore, the Grothendieck model functor maps the original pushout square of signatures in the Grothendieck institution to a pullback square by composing the four pullback squares obtained from mapping the four component pushout squares.

Unlike the corresponding results for theory colimits or liberality, the result of Theorem 5 cannot always be applied in practice; there are important practical cases when the necessary conditions for the (semi-)exactness of the Grothendieck institution do not hold. In such situations one should try to base the semantics of the specification language on a subclass of practically meaningful cases for which the (semi-)exactness property can be obtained (Diaconescu, 1998). In this case Theorem 5

[6] Notice that in the diagrams of this proof we represent the indices by their corresponding signature categories and the index morphisms by the corresponding functors between the signature categories.

allow us to isolate the condition which is responsible for the failure of the (semi-)exactness property. It seems that in practice only the last two conditions of Theorem 5 might fail to hold.

4.4. Inclusion Systems in Grothendieck Institutions

Inclusion systems where first introduced by (Diaconescu et al., 1993) for the institution-independent study of structuring specifications. They provide the underlying mathematical concept for module imports, which are the most fundamental structuring construct. Mathematically, inclusion systems capture categorically the concept of set-theoretic 'inclusion' in a way reminiscent of factorization systems (Borceux, 1994). *Weak inclusion systems* were introduced in (Căzănescu and Roşu, 1997) as a weakening of the original definition of inclusion systems of (Diaconescu et al., 1993).

DEFINITION 16. $\langle \mathcal{I}, \mathcal{E} \rangle$ is a *weak inclusion system* for a category $\mathbb{C}$ if $\mathcal{I}$ and $\mathcal{E}$ are two subcategories with $|\mathcal{I}| = |\mathcal{E}| = |\mathbb{C}|$ such that

1. $\mathcal{I}$ is a partial order, and
2. every arrow f in $\mathbb{C}$ can be factored uniquely as $f = e;i$ with $e \in \mathcal{E}$ and $i \in \mathcal{I}$.

The arrows of $\mathcal{I}$ are called *inclusions*, and the arrows of $\mathcal{E}$ are called *surjections*.[7] The domain (source) of the inclusion i in the factorization of f is called called the *image of* f and denoted as $\mathrm{Im}(f)$. An *injection* is a composite between an inclusion and an isomorphim.

A weak inclusion system $\langle \mathcal{I}, \mathcal{E} \rangle$ is an *inclusion system* if and only if $\mathcal{I}$ has finite least upper bounds (denoted $+$) and all surjections are epics (see (Diaconescu et al., 1993)). □

Recall from (Diaconescu, 1998) the following technical definition:

DEFINITION 17. Let $\mathbb{C}$ and $\mathbb{C}'$ be two categories with weak inclusion systems $\langle \mathcal{I}, \mathcal{E} \rangle$ and $\langle \mathcal{I}', \mathcal{E}' \rangle$ respectively. Then a functor $\mathcal{U}\colon \mathbb{C} \to \mathbb{C}'$ *lifts inclusions uniquely* if and only if for any inclusion $\iota' \colon A' \hookrightarrow B\mathcal{U}$ in $\mathcal{I}'$ with $B \in |\mathbb{C}|$, there exists a unique inclusion $\iota \in \mathcal{I}$ such that $\iota\mathcal{U} = \iota'$. □

Because of the structure of the Grothendieck institutions (see Remark 1), the problem of an inclusion system for its category of signatures is reduced to the problem of inclusion systems in Grothendieck categories. However, in this paper we limit this study to the case of weak inclusion systems.

4.4.1. *Inclusion Systems in Grothendieck categories*

THEOREM 6. *Let $B\colon I^{\mathrm{op}} \to \mathbb{C}at$ be an indexed category such that*

- *I has a weak inclusion system $\langle \mathcal{I}^I, \mathcal{E}^I \rangle$,*
- *B^i has a weak inclusion system $\langle \mathcal{I}^i, \mathcal{E}^i \rangle$ for each index $i \in |I|$,*
- *B^u preserves inclusions for each inclusion index morphism $u \in \mathcal{I}^I$, and*
- *B^u preserves inclusions and surjections and lifts inclusions uniquely for each surjection index morphism $u \in \mathcal{E}^I$.*

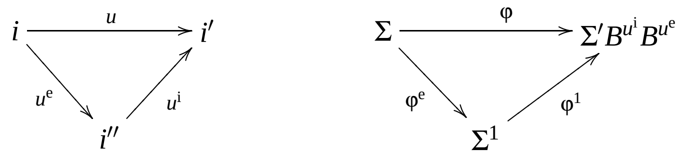
Then, the Grothendieck category $B^\sharp$ has an inclusion system $\langle \mathcal{I}^{B^\sharp}, \mathcal{E}^{B^\sharp} \rangle$ where $\langle u, \varphi \rangle$ is

- inclusion *iff both u and φ are inclusions, and*
- surjection *iff both u and φ are surjections.*

[7] Surjections of some weak inclusion systems need not necessarily be surjective in the ordinary sense.

Proof. $I^{B^\sharp}$ and $\mathcal{E}^{B^\sharp}$ are both sub-categories of $B^\sharp$ because B^u preserves inclusions (surjections) whenever u is inclusion (surjection).

We now consider an arrow $\langle u, \varphi\rangle : \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$ in the Grothendieck category $B^\sharp$ and prove that it factors uniquely as a composite between an arrow from $\mathcal{E}^{B^\sharp}$ and an arrow from $I^{B^\sharp}$. We factor u in $\langle I^I, \mathcal{E}^I\rangle$ and φ in $\langle I^i, \mathcal{E}^i\rangle$ as follows:



Since u^{e} lifts inclusions uniquely there exists an unique inclusion $\varphi^{\mathrm{i}} : \Sigma'' \to \Sigma' B^{u^{\mathrm{i}}}$ such that $\varphi^{\mathrm{i}} B^{u^{\mathrm{e}}} = \varphi^1$. Notice that $\langle u^{\mathrm{i}}, \varphi^{\mathrm{i}}\rangle$ is an inclusion, $\langle u^{\mathrm{e}}, \varphi^{\mathrm{e}}\rangle$ is a surjection, and that $\langle u, \varphi\rangle = \langle u^{\mathrm{e}}, \varphi^{\mathrm{e}}\rangle; \langle u^{\mathrm{i}}, \varphi^{\mathrm{i}}\rangle$.

Finally, the uniqueness of this factorization follows stepwise from the uniqueness of the factorization of the index morphism, then from the uniqueness of the factorization through the inclusion system of B^i (by using the preservation of inclusions by the $B^{u^{\mathrm{e}}}$), and finally from the uniqueness of the lifting to $I^{i''}$.

A similar result was proved in (Diaconescu, 1998) directly for extra theory morphisms. Theorem 6 avoids some complexities of the corresponding result from (Diaconescu, 1998) which were related to the sentences. This simplification is possible due to the fact that we have a (Grothendieck) institution in which extra theory morphisms appear as ordinary theory morphisms which permits the automatic lifting of inclusion systems from signatures to theories (see (Diaconescu et al., 1993; Căzănescu and Roşu, 1997)). In fact, as pointed out by (Căzănescu and Roşu, 1997), for the case of weak inclusion systems this can be done in two different ways, thus obtaining two different weak inclusion systems at the level of theories for each weak inclusion system for signatures. In this way, from Theorem 6 one can also obtain a different variant of the result in (Diaconescu, 1998) corresponding to the other way of lifting of weak inclusion systems from signatures to theories, which shows that Theorem 6 is conceptually more general than the result of (Diaconescu, 1998).

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5. Conclusions

We extended the concepts of Grothendieck and fibred categories to institutions, including a Grothendieck construction for institutions (easily extensible to other related structures) and an equivalence result á la Bénabou between Grothendieck and fibred institutions. We showed that the concept of extra theory morphism of (Diaconescu, 1998) appears as ordinary (intra) theory morphism in a

Grothendieck institution, leading to a higher conceptual approach to multi-paradigm (heterogenous) algebraic specification. We also extended the 'globalization' results of institutional properties of (Diaconescu, 1998), by giving necessary and sufficient conditions for theory colimits, liberality, exactness in Grothendieck institutions, and by providing inclusion systems to Grothendieck categories. The conceptual power of Grothendieck institutions enabled us to extend the results of (Diaconescu, 1998), also by highly simplifying some of the proofs, and to give a necessary and sufficient characterization for the exactness problem in Grothendieck institutions (conjectured in (Diaconescu, 1998)).

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