Extra Theory Morphisms for Institutions:
logical semantics for multi-paradigm languages

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Abstract

We extend the ordinary concept of theory morphism in institutions to *extra* theory morphisms. Extra theory morphism map theories belonging to different institutions across institution morphisms. We investigate the basic mathematical properties of extra theory morphisms supporting the semantics of logical multi-paradigm languages, especially structuring specifications (module systems) á la OBJ-Clear. They include model reducts, free constructions (liberality), co-limits, model amalgamation (exactness), and inclusion systems.

We outline a general logical semantics for languages whose semantics satisfy certain "logical" principles by extending the institutional semantics developed within the Clear-OBJ tradition. Finally, in the Appendix, we briefly illustrate it with the concrete example of CafeOBJ.

Keywords

Algebraic specification, Institutions, Theory morphism.

AMS Classifications

68Q65, 18C10, 03G30, 08A70

1 Introduction

Computing Motivation

This work belongs to the research area of *institutions* which now constitute the modern level of algebraic specification tradition. Its results apply to multi-paradigm *logical* computing languages. A **logical** language is a specification and/or programming language having an underlying logic[1] in which all its basic constructs/features can be rigorously explained. This concept was first formulated by Goguen and Meseguer in [22] under the name of "logical programming". Examples of logical languages include most of the OBJ family of languages, such as OBJ3 [25], Eqlog [21], Maude [5], CafeOBJ [13], etc., but they might also include (pure) Prolog and (pure) Lisp.

Multi-paradigm logical languages admit institution semantics in which each paradigm has an underlying institution and paradigm embedding formally corresponds to institution morphisms. This approach can be regarded as *relativistic* as opposed to the more absolute one that works only with one big institution embedding all other institutions underlying the various paradigms, and has also been advocated by other recent works [32, 2]. Following the tradition of Clear [4] and OBJ [25], flattened modules or basic specifications (belonging to either a primary or a more complex paradigm) in logical languages correspond to theories in the institution underlying that paradigm. A logical language achieving high paradigm integration must support a global module system, meaning global structuring operations on modules (specifications), such as (various kinds of) imports, parameterization, etc. As strongly emphasized by the Clear-OBJ tradition, the structuring operations on modules are modeled by putting together theories via co-limits of theory morphisms.

In this paper we extend the concept of theory morphism (traditionally local to a given institution; we call them *intra* theory morphisms) to *extra* theory morphisms, which are morphisms of theories across institution morphisms (embeddings). The core of this work consists of the investigation of the basic properties for extra theory morphisms supporting the semantics of multi-paradigm logical languages, especially advanced module systems for such languages. We devote a section to sketching an extra theory morphism based generic semantics for multi-paradigm logical languages, and in the Appendix we illustrate this with the concrete example of CafeOBJ [14].

Properties of Extra Theory Morphisms

The basic mathematical properties of theory morphisms are well established for the ordinary "intra" version, in this paper we investigate them for the more general "extra" concept. Here we briefly review these properties.

Liberality. Liberality [17, 31] is a basic desirable property expressing the possibility of free constructions generalizing the principle of "initial algebra semantics" which underlies the tight semantics of algebraic languages, including semantics for parameterized modules [15]. We extend the traditional concept of liberality to extra theory morphisms and we investigate some natural sufficient conditions.

Co-limits. Module expressions in algebraic languages in the Clear-OBJ tradition are evaluated as co-limits of theories. In the case of multi-paradigm languages, co-limits in the category of extra theory morphisms are needed. We show how these more general co-limits can be constructed from ordinary intra theory morphisms co-limits.

Exactness. Exactness expresses the possibility of amalgamation of consistent implementations for different modules (for more details see [15]) and is a necessary technical condition on the underlying logic

[1] Here "logic" should be understood in the modern relativistic sense of "institution" which provides a mathematical definition for a logic (see [17]) rather than in the traditional sense.

for good semantic properties of the module system. We study exactness properties for the general case of extra theory morphisms.

Inclusions. Theory inclusions model mathematically the concept of module import (see [15]), which is the most fundamental structuring operation. *Inclusion systems* where first introduced in [15] as the underlying categorical structure of an institution-independent module algebra. They were further studied and their definition simplified by Roşu and Căzănescu in [6]. Inclusion systems are related to the better established concept of factorization systems, but they capture the uniqueness property of inclusions (such as set-theoretic inclusions). Here we show that an inclusion system of institution morphisms together with inclusion systems for the signatures of each of the institutions involved naturally determine an inclusion system for extra theory morphisms.

Finally, this work assumes familiarity with the basics of category theory, and generally uses the same notation as Mac Lane [27], except that composition is denoted by ";" and written in the diagrammatic order. The application of functions (functors) to arguments may be written either normally using parentheses, or else in diagrammatic order without parentheses. The category of sets is denoted as $\mathbb{S}et$, and the category of categories[2] as $\mathbb{C}at$. The opposite of a category $\mathbb{C}$ is denoted by $\mathbb{C}^{op}$. The class of objects of a category $\mathbb{C}$ is denoted by $|\mathbb{C}|$; also the set of arrows in $\mathbb{C}$ having the object a as source and the object b as target is denoted as $\mathbb{C}(a,b)$.

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2 Institutions

In this section we start by reviewing some of the basic concepts and results on institutions, and continue with introducing new concepts related to extra theory morphisms. A good introduction to institutions is [17], and [15] contains many results about institutions with direct application to modularization.

From a logic perspective, institutions are much more abstract than Tarski's model theory, and also have another basic ingredient, namely signatures and the possibility of translating sentences and models across signature morphisms. A special case of this translation is familiar in first order model theory: if $\Sigma \to \Sigma'$ is an inclusion of first order signatures and M is a Σ'-model, then we can form the *reduct* of M to Σ, denoted $M{\restriction}_{\Sigma}$. Similarly, if e is a Σ-sentence, we can always view it as a Σ'-sentence (but there is no standard notation for this). The key axiom, called the **satisfaction condition**, says that *truth is invariant under change of notation*, which is surely a very basic intuition for traditional logic.

Definition 1: An **institution** $\mathfrak{I} = (\mathbb{S}ign, Sen, \mathrm{MOD}, \models)$ consists of

1. a category $\mathbb{S}ign$, whose objects are called **signatures**,
2. a functor $Sen\colon \mathbb{S}ign \to \mathbb{S}et$, giving for each signature a set whose elements are called **sentences** over that signature,
3. a functor $\mathrm{MOD}\colon \mathbb{S}ign \to \mathbb{C}at^{op}$ giving for each signature Σ a category whose objects are called **Σ-models**, and whose arrows are called **Σ-(model) morphisms**, and

[2] We stay away of any foundational problem related to the "category of all categories"; several solutions can be found in the literature, see for example [27].

4. a relation $\models_\Sigma \subseteq |\mathrm{MOD}(\Sigma)| \times Sen(\Sigma)$ for each $\Sigma \in |\mathbb{S}ign|$, called **$\Sigma$-satisfaction**,

such that for each morphism $\varphi\colon \Sigma \to \Sigma'$ in $\mathbb{S}ign$, the **satisfaction condition**

$$M' \models_{\Sigma'} Sen(\varphi)(e) \ \text{ iff } \ \mathrm{MOD}(\varphi)(M') \models_\Sigma e$$

holds for each $M' \in |\mathrm{MOD}(\Sigma')|$ and $e \in Sen(\Sigma)$. We may denote the reduct functor $\mathrm{MOD}(\varphi)$ by $\_\restriction_\varphi$ and the sentence translation $Sen(\varphi)$ by $\varphi(\_)$. $\boxed{1}$

Definition 2: Let $\mathfrak{I} = (\mathbb{S}ign, Sen, \mathrm{MOD}, \models)$ be an institution. For any signature Σ the closure of a set E of Σ-sentences is $E^\bullet = \{e \mid E \models_\Sigma e\}$[3]. (Σ, E) is a **theory** iff $E = E^\bullet$.

A **theory morphism** $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ is a signature morphism $\varphi\colon \Sigma \to \Sigma'$ such that $\varphi(E) \subseteq E'$. Let $\mathbb{T}h(\mathfrak{I})$ denote the category of all theories in $\mathfrak{I}$, and $sign^{\mathfrak{I}}$ the forgetful functor $\mathbb{T}h(\mathfrak{I}) \to \mathbb{S}ign$. $\boxed{2}$

For any institution $\mathfrak{I}$, the model functor MOD extends to $\mathbb{T}h(\mathfrak{I})$, by mapping a theory (Σ, E) to the full subcategory $\mathrm{MOD}(\Sigma, E)$ of $\mathrm{MOD}(\Sigma)$ formed by the Σ-models that satisfy E.

Definition 3: A theory morphism $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ is **liberal** iff the reduct functor $\_\restriction_\varphi\colon \mathrm{MOD}(\Sigma', E') \to \mathrm{MOD}(\Sigma, E)$ has a left-adjoint $(\_)^\varphi$.

$$\begin{tikzcd} M \models_\Sigma E & M \arrow[r] \arrow[d, "h"'] & (M^\varphi)\restriction_\varphi \arrow[dl, "h'\restriction_\varphi"] & M^\varphi \arrow[dl, "\text{there exists a unique } h'"] \\ M' \models_{\Sigma'} E' & M'\restriction_\varphi & M' & \end{tikzcd}$$

The institution $\mathfrak{I}$ is **liberal** iff each theory morphism is liberal. When (Σ, E) is the empty theory for the signature Σ, we denote $(\_)^\varphi$ by $\_/E'$. $\boxed{3}$

General results [31] show that liberality is equivalent to the power of Horn axiomatisability.

Definition 4: An institution $\mathfrak{I}$ is **exact** iff the model functor $\mathrm{MOD}\colon \mathbb{S}ign \to \mathbb{C}at^{op}$ preserves finite colimits. $\mathfrak{I}$ is **semi-exact** iff MOD preserves only pushouts. $\boxed{4}$

The possibility of amalgamating consistent implementations may also be formalized by a weak[4] version of exactness, which in the case of multi-paradigm languages is more adequate.

Definition 5: An institution $\mathfrak{I}$ is **weakly semi-exact** iff the model functor MOD preserves weak pushouts. $\boxed{5}$

Semantics of multi-paradigm systems involves several different institutions which have to be linked together by using the following concept:

Definition 6: Let $\mathfrak{I}$ and $\mathfrak{I}'$ be institutions. Then an **institution morphism** $\mathfrak{I} \to \mathfrak{I}'$ consists of

1. a functor $\Phi\colon \mathbb{S}ign' \to \mathbb{S}ign$,
2. a natural transformation $\alpha\colon \Phi; Sen \Rightarrow Sen'$, and
3. a natural transformation $\beta\colon \Phi; \mathrm{MOD} \Rightarrow \mathrm{MOD}'$

[3] $E \models_\Sigma e$ means that $M \models_\Sigma e$ for any Σ-model M that satisfies all sentences in E.

[4] In the sense of "weak universal properties" of [27] requiring only *existence* without uniqueness for the corresponding universal arrows.

such that the following **satisfaction condition** holds

$$M' \models_{\Sigma'} \alpha_{\Sigma'}(e) \ \text{ iff } \ \beta_{\Sigma'}(M') \models'_{\Sigma'\Phi} e$$

for any Σ'-model M' from $\Im'$ and any $\Sigma'\Phi$-sentence e from $\Im$. $\boxed{6}$

In the literature there are several concepts of institution morphism, each of them being adequate to some specific problem. A good survey of various concepts of institution morphism discussing their usefulness can be found in [32]. The definition presented above and originally given by Goguen and Burstall [17] seems to be the most adequate for our approach. However, in our paper the direction of the institution morphisms goes opposite than in [17]. The choice of [17] (favored by many researchers) fits better with the understanding of institution morphisms as projections [2], while our choice is motivated mainly by the co-limit and especially the inclusion paradigms.

For obtaining some technical properties for extra theory morphisms, some technically stronger versions of this institution morphism are needed. These are very natural technical conditions which are easily satisfied in practice.

Definition 7: An institution morphism $(\Phi, \alpha, \beta) \colon \Im \to \Im'$ is

- a **[strong] embedding** iff Φ admits a [left-inverse] left-adjoint [with identity units] $\overline{\Phi}$,
- **liberal** iff $\beta_{\Sigma'}$ has a left-adjoint $\overline{\beta}_{\Sigma'}$ for each $\Sigma' \in |\mathbb{S}ign'|$, and **persistent** iff in addition $\overline{\beta}_{\Sigma'}$ are also left-inverses to $\beta_{\Sigma'}$ with identity units, and
- **[weakly] additive** iff the squares defining the naturality of β are [weak] pullbacks.

$$\begin{tikzcd}
\mathrm{Mod}(\Sigma'\Phi) & \mathrm{Mod}'(\Sigma') \arrow[l, "\beta_{\Sigma'}"'] \\
\mathrm{Mod}(\Sigma'_1\Phi) \arrow[u, "\mathrm{Mod}(\varphi\Phi)"] & \mathrm{Mod}'(\Sigma'_1) \arrow[l, "\beta_{\Sigma'_1}"] \arrow[u, "\mathrm{Mod}'(\varphi)"']
\end{tikzcd}$$

$\boxed{7}$

The idea of institution embedding (although not formulated directly) is as old as the seminal work on institutions [17]. Notice that the terminology "institution embedding" is used also by Meseguer [29] but in a completely different sense. Also, several variants of persistent institution morphism have been independently introduced in the literature, such that the *categorical retractive simulations* of [26] and the *extension maps* of [29].

3 Extra Theory Morphisms

Extra theory morphisms generalize the ordinary concept of theory morphism (Definition 2) in that they map theories across an institution morphism. Intra (i.e., ordinary) theory morphisms can be regarded as special cases when the institution morphism is an identity.

Definition 8: Let $(\Phi, \alpha, \beta) \colon \Im \to \Im'$ be an institution morphism, and $T = (\Sigma, E)$ and $T' = (\Sigma', E')$ be theories in $\Im$ and $\Im'$ respectively. A **extra theory morphism** $T \to T'$ is an $\Im$-signature morphism $\varphi \colon \Sigma \to \Sigma'\Phi$ such that $\alpha_{\Sigma'}(\varphi(E)) \subseteq E'$. $\boxed{8}$

In the case of institution embeddings we have an equivalent simpler formulation for extra theory morphism given by Proposition 10. Instances of the following result for the particular case of strong embeddings appeared in [17, 2]:

Fact 9: Any institution embedding $(\Phi,\alpha,\beta)\colon \mathfrak{I} \to \mathfrak{I}'$ gives rise to a functor $\Phi^*\colon \mathbb{T}h(\mathfrak{I}) \to \mathbb{T}h(\mathfrak{I}')$ defined by

$$\Phi^*(\Sigma,E) = (\Sigma\overline{\Phi}, \alpha_{\Sigma\overline{\Phi}}((\Sigma\zeta)(E))^\bullet)$$

where ζ is the unit of the adjoint pair of functors $\Phi, \overline{\Phi}$. $\boxed{9}$

Proposition 10: Let $(\Phi,\alpha,\beta)\colon \mathfrak{I} \to \mathfrak{I}'$ be an institution embedding and let $T \in |\mathbb{T}h(\mathfrak{I})|$ and $T' \in |\mathbb{T}h(\mathfrak{I}')|$. Then there is a natural bijection between extra theory morphisms $T \to T'$ and $\mathfrak{I}'$-theory morphisms $\Phi^*(T) \to T'$.

Proof: Let T be (Σ,E) and T' be (Σ',E'). We have to establish a canonical bijection between extra theory morphisms $T \to T'$ and $\mathfrak{I}'$-theory morphisms $\Phi^*(T) \to T'$. This is given by the restriction of the adjointness bijection

$$\mathbb{S}ign(\Sigma, \Sigma'\Phi) \overset{\gamma}{\simeq} \mathbb{S}ign'(\Sigma\overline{\Phi}, \Sigma')$$

to the subsets $\{\varphi \mid \alpha_{\Sigma'}(\varphi(E)) \subseteq E'\}$ and $\{\varphi' \mid \varphi'(\alpha_{\Sigma\overline{\Phi}}((\Sigma\zeta)(E)) \subseteq E'\}$. This is well-defined because whenever $\varphi' = \gamma(\varphi)$, $\alpha_{\Sigma'}(\varphi(E)) = \varphi'(\alpha_{\Sigma\overline{\Phi}}((\Sigma\zeta)(E))$ by the naturality of α and because $\overline{\Phi}$ is a left-adjoint to Φ, as shown in the following diagram:

$$\begin{tikzcd}
 & Sen(\Sigma'\Phi) \arrow[r, "\alpha_{\Sigma'}"] & Sen'(\Sigma') \\
 & Sen(\Sigma\overline{\Phi}\Phi) \arrow[u, "Sen(\varphi'\Phi)"'] \arrow[r, "\alpha_{\Sigma\overline{\Phi}}"] & Sen'(\Sigma\overline{\Phi}) \arrow[u, "Sen'(\varphi')"'] \\
Sen(\Sigma) \arrow[uur, "Sen(\varphi)"] \arrow[ur, "Sen(\Sigma\zeta)"'] & &
\end{tikzcd}$$

$\boxed{10}$

For readers familiar with indexed categories [33], the previous results just says that in the case of institution embeddings extra theory morphisms can be regarded as arrows in the flattening (i.e., the Grothendick construction) of the indexed (by the category of institutions) category $\mathbb{T}h$.

3.1 Model Reducts

Model reducts are the semantic aspect of theory morphisms, therefore they play a central rôle in any semantics based on institutions. Model reducts for extra theory morphisms generalize ordinary model reducts for intra theory morphisms; they are introduced by the following result which can also be regarded as a Satisfaction Condition for extra theory morphisms:

Proposition 11: Let $(\Phi,\alpha,\beta)\colon \mathfrak{I} \to \mathfrak{I}'$ be an institution morphism. For any extra theory morphism $\varphi\colon (\Sigma,E) \to (\Sigma',E')$ there is a reduct functor $\_\restriction_\varphi\colon \mathrm{M{\scriptsize OD}}'(T') \to \mathrm{M{\scriptsize OD}}(T)$ defined by

$$M'\restriction_\varphi = \beta_{\Sigma'}(M')\restriction_\varphi$$

for M' any (Σ',E')-model. If (Φ,α,β) is an embedding, then

$$M'\restriction_\varphi = \beta_{\Sigma\overline{\Phi}}(M'\restriction_{\varphi'})\restriction_{\Sigma\zeta}$$

where $\varphi'\colon \Sigma\overline{\Phi} \to \Sigma'$ is the free extension of $\varphi\colon \Sigma \to \Sigma'\Phi$.

Proof: Firstly, we have to prove that $M' \models_{\Sigma'} E'$ implies $\beta_{\Sigma'}(M')\restriction_\varphi \models_\Sigma E$. But $M' \models_{\Sigma'} E'$ implies $M' \models_{\Sigma'} \alpha_{\Sigma'}(\varphi(E))$ which is equivalent (by the Satisfaction Condition for institution morphisms) to $\beta_{\Sigma'}(M') \models_{\Phi(\Sigma')} \varphi(E)$ which is equivalent (by the Satisfaction Condition for institutions) to $\beta_{\Sigma'}(M')\restriction_\varphi \models_\Sigma E$.

The second part follows by the naturality of β applied to φ and because $\overline{\Phi}$ is a left-adjoint to Φ, as shown in the following diagram:

$$\begin{tikzcd}
\mathrm{MOD}'(\Sigma') \arrow[r, "\beta_{\Sigma'}"] \arrow[d, "\mathrm{MOD}'(\varphi')"'] & \mathrm{MOD}(\Sigma'\Phi) \arrow[d, "\mathrm{MOD}(\varphi'\Phi)"'] \arrow[ddr, "\mathrm{MOD}(\varphi)"] & \\
\mathrm{MOD}'(\Sigma\overline{\Phi}) \arrow[r, "\beta_{\Sigma\overline{\Phi}}"'] & \mathrm{MOD}(\Sigma\overline{\Phi}\Phi) \arrow[dr, "\mathrm{MOD}(\Sigma\zeta)"'] & \\
& & \mathrm{MOD}(\Sigma)
\end{tikzcd}$$

$\boxed{11}$

3.2 Liberality

Liberality (i.e., the existence of free extensions along theory morphisms) is one of the fundamental model-theoretic properties in strong connection with module system semantics for specification/programming multi-paradigm logical languages.

In this section we provide sufficient conditions for the existence of free extensions along extra theory morphisms. We fix an institution morphism $(\Phi,\alpha,\beta)\colon \mathfrak{I} \to \mathfrak{I}'$.

Definition 12: An extra theory morphism $\varphi\colon (\Sigma,E) \to (\Sigma',E')$ is **liberal** iff the reduct functor $\_\restriction_{\varphi}\colon \mathrm{MOD}'(\Sigma',E') \to \mathrm{MOD}(\Sigma,E)$ has a left adjoint, i.e., iff for any model $M \in |\mathrm{MOD}(\Sigma,E)|$, there exists a model $M' \in |\mathrm{MOD}'(\Sigma',E')|$ and a model morphism $M\eta\colon M \to M'\restriction_{\varphi}$ such that for any model $N' \in |\mathrm{MOD}'(\Sigma',E')|$ and any model morphism $h\colon M \to N'\restriction_{\varphi}$ there exists a unique model morphism $h'\colon M' \to N'$ such that $h = M\eta; h'\restriction_{\varphi}$.

$$\begin{tikzcd}
M \arrow[r, "M\eta"] \arrow[dr, "h"'] & M'\restriction_{\varphi} \arrow[d, "h'\restriction_{\varphi}"] & M' \arrow[d, "h'"] \\
& N'\restriction_{\varphi} & N'
\end{tikzcd}$$

$\boxed{12}$

We need a categorical lemma:

Lemma 13: Let $\mathcal{U}\colon \mathbb{C} \to \mathbb{D}$ be a functor with a left-adjoint $\mathcal{F}$, let $\mathbb{C}' \hookrightarrow \mathbb{C}$ be a full reflective subcategory, and $\mathbb{D}' \hookrightarrow \mathbb{D}$ be a full subcategory, such that $\mathcal{U}(\mathbb{C}') \subseteq \mathbb{D}'$. Then the restriction $\mathcal{U}\restriction_{\mathbb{C}'}\colon \mathbb{C}' \to \mathbb{D}'$ has a left-adjoint.

Proof: Let us denote the left-adjoint to $\mathbb{C}' \hookrightarrow \mathbb{C}$ by $\mathcal{R}$. Then for each $d' \in |\mathbb{D}'|$ and $c' \in |\mathbb{C}'|$, we have the following natural isomorphisms: $\mathbb{D}'(d', c'\mathcal{U}) \simeq \mathbb{D}(d', c'\mathcal{U}) \simeq \mathbb{C}(d'\mathcal{F}, c') \simeq \mathbb{C}'(d'\mathcal{F}\mathcal{R}, c')$. $\boxed{13}$

Theorem 14: If $\mathfrak{I}$ is liberal on signature morphisms, $\mathfrak{I}'$ and (Φ,α,β) are liberal, then any extra theory morphism $\varphi\colon (\Sigma,E) \to (\Sigma',E')$ is liberal. Moreover, the free (Σ',E')-model over a given (Σ,E)-model M can be obtained as $\overline{\beta}_{\Sigma'}(M^{\varphi})/E'$.

Proof: This is obtained by applying Lemma 13 for $\mathcal{U}$ the composite of the two right-adjoint functors

$$\mathrm{MOD}'(\Sigma') \xrightarrow{\beta_{\Sigma'}} \mathrm{MOD}(\Sigma'\Phi) \xrightarrow{-\restriction_{\varphi}} \mathrm{MOD}(\Sigma)$$

and for the full subcategory $\mathrm{MOD}(\Sigma,E) \hookrightarrow \mathrm{MOD}(\Sigma)$ and the full reflective subcategory $\mathrm{MOD}'(\Sigma',E') \hookrightarrow \mathrm{MOD}'(\Sigma')$. The condition $\mathcal{U}(\mathrm{MOD}'(\Sigma',E')) \subseteq \mathrm{MOD}(\Sigma,E)$ holds by Proposition 11. $\boxed{14}$

In the case of institution embeddings we can obtain another sufficient condition for the liberality of extra theory morphisms without requiring any liberality for $\Im$.

Theorem 15: If (Φ,α,β) is a strong liberal embedding, $\Im'$ is liberal, and $\overline{\beta}$ satisfies the following Satisfaction Condition:

$$\overline{\beta}_{\Sigma'}(M) \models_{\Sigma'} \alpha_{\Sigma'}(e) \text{ if } M \models_{\Sigma'\Phi} e$$

for all $M \in |\mathrm{MOD}(\Sigma'\Phi)|$ and $e \in Sen(\Sigma'\Phi)$, then each extra theory morphism $\varphi\colon (\Sigma,E) \to (\Sigma',E')$ is liberal.

Proof: Following the second part of Proposition 11, the reduct $\mathrm{MOD}'(\Sigma',E') \xrightarrow{-\restriction_\varphi} \mathrm{MOD}(\Sigma,E)$ can be factored as

$$\mathrm{MOD}'(\Sigma',E') \xrightarrow{-\restriction_{\varphi'}} \mathrm{MOD}'(\Sigma\overline{\Phi},\alpha_{\Sigma\overline{\Phi}}(E)^\bullet) \xrightarrow{\beta_{\Sigma\overline{\Phi}}} \mathrm{MOD}(\Sigma,E)$$

The first reduct has a left-adjoint because $\Im'$ is liberal, and the second has as left-adjoint the restriction of $\overline{\beta}_{\Sigma\overline{\Phi}}$ to $\mathrm{MOD}(\Sigma,E)$. This works well because if $M \in |\mathrm{MOD}(\Sigma,E)|$, then $\overline{\beta}_{\Sigma\overline{\Phi}}(M) \models_{\Sigma\overline{\Phi}} \alpha_{\Sigma\overline{\Phi}}(e)$. $\boxed{15}$

Corollary 16: If (Φ,α,β) is a persistent strong embedding and $\Im'$ is liberal, then any extra theory morphism is liberal.

Proof: Notice that the Satisfaction Condition from the hypotheses of Theorem 15 follows from the Satisfaction Condition for persistent institution morphisms. $\boxed{16}$

3.3 Theory Co-limits

Co-limits of theories are the main technical tool for evaluating module expressions in the OBJ-Clear tradition. In the case of multi-paradigm languages one has to consider extra theory morphisms for computing such co-limits.

In this section we study co-limits of extra theory morphisms. The co-limit of a diagram of extra theory morphisms is computed in a pre-defined fixed institution in which all institutions underlying the nodes of the diagram are embedded. This is more general than just doing it in the co-limit of the underlying diagram of institution embeddings[5]; since in applications the co-cones of the underlying institutions are not necessarily co-limit co-cones.[6]

For the purpose of this section we fix a diagram of institution morphisms $(\Im_i)_{i\in J}$, where $\Im_i = (\mathbb{S}ign^i, Sen^i, \mathrm{MOD}^i, \models^i)$ are institutions for $i \in |J|$ and $\Im_u = (\Phi^u,\alpha^u,\beta^u)\colon \Im_i \to \Im_j$ are institution morphisms for all $u \in J(i,j)$.

Then we fix a co-cone of institution embeddings $(\Phi^i,\alpha^i,\beta^i)_{i\in|J|}\colon (\Im_i)_{i\in J} \to \Im_0$.

Theorem 17: Given a diagram $(T_i)_{i\in J}$ of extra theory morphisms such that

- $T_i = (\Sigma_i,E_i)$ is a theory in $\Im_i$ for each $i \in |J|$, and
- $T_u = \varphi^u$ is a extra theory morphism $T_i \to T_j$ with $(\Phi^u,\alpha^u,\beta^u)$ the underlying institution morphism, for all $u \in J(i,j)$,

and assuming that $\mathbb{S}ign_0$ has J-co-limits, then there exists a theory $T_0 = (\Sigma_0,E_0)$ in $\Im_0$ and a co-cone

$$(\varphi^i)_{i\in|J|}\colon (T_i)_{i\in J} \to T_0$$

with $(\Phi^i,\alpha^i,\beta^i)_{i\in|J|}$ the underlying co-cone of institution morphisms, such that given any other institution morphism $(\Phi^0,\alpha^0,\beta^0)\colon \Im_0 \to \Im_1$ and any extra theory morphism co-cone $(\theta^i)_{i\in|J|}\colon (T_i)_{i\in J} \to T_1$ with $T_1 = (\Sigma_1,E_1)$ and $((\Phi^i,\alpha^i,\beta^i);(\Phi^0,\alpha^0,\beta^0))_{i\in|J|}$ being the underlying co-cone of institution morphisms for $(\theta^i)_{i\in|J|}$, then there exists an unique extra theory morphism $\delta\colon T_0 \to T_1$ such that $\varphi^i;\delta = \theta^i$ for all $i \in |J|$.

[5]This co-limit exists by the fundamental results on existence of co-limits of institutions of [17].
[6]This is also the case of CafeOBJ see the Appendix.

$$
\begin{tikzcd}
\Im_i \arrow[dd, "{(\Phi^u,\alpha^u,\beta^u)}"'] \arrow[dr, "{(\Phi^i,\alpha^i,\beta^i)}"] & & \\
& \Im_0 \arrow[r, "{(\Phi^0,\alpha^0,\beta^0)}"] & \Im_1 \\
\Im_j \arrow[ur, "{(\Phi^j,\alpha^j,\beta^j)}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\Sigma_i,E_i) \arrow[dd, "\varphi^u"'] \arrow[dr, "\varphi^i"'] \arrow[drr, "\theta^i"] & & \\
& (\Sigma_0,E_0) \arrow[r, "\delta"] & (\Sigma_1,E_1) \\
(\Sigma_j,E_j) \arrow[ur, "\varphi^j"] \arrow[urr, "\theta^j"'] & &
\end{tikzcd}
$$

Proof: The plan of this proof is as follows:

1. The J-diagram $(T_i)_{i\in J}$ of extra theory morphisms generates a J-diagram $(\mathbf{T}_i)_{i\in J}$ of intra theory morphisms in $\Im'$, where $\mathbf{T}_i = \Phi^{*i}(T_i)$ for all $i \in |J|$,
2. Let (Σ_0, E_0) be the co-limit of $(\mathbf{T}_i)_{i\in J}$ with $(\Sigma_i\overline{\Phi^i} \xrightarrow{(\varphi^i)'} \Sigma_0)_{i\in|J|}$ the co-limit co-cone. Then $(\Sigma_i \xrightarrow{\varphi^i} \Sigma_0\Phi^i)_{i\in|J|}$ is the corresponding co-cone $(T_i)_{i\in J} \to T_0$.
3. Given $(\theta^i)_{i\in|J|}$ we prove the existence and uniqueness of $\delta\colon T_0 \to T_1$ such that $\varphi^i;\delta = \theta^i$ for all $i \in |J|$.

1. Each arrow $\varphi^u\colon (\Sigma_i, E_i) \to (\Sigma_j, E_j)$ in the original diagram of extra theory morphisms gets mapped to the theory morphism

$$(\varphi^u;\Phi^u(\Sigma_j\zeta^j))'\colon (\Sigma_i\overline{\Phi^i}, \alpha^i_{\Sigma_i\overline{\Phi^i}}((\Sigma_i\zeta^i)(E_i))^\bullet) \to (\Sigma_j\overline{\Phi^j}, \alpha^j_{\Sigma_j\overline{\Phi^j}}((\Sigma_j\zeta^j)(E_j))^\bullet)$$

(see the following diagram)

$$
\begin{tikzcd}
\Sigma_i \arrow[r, "\Sigma_i\zeta^i"] \arrow[d, "\varphi^u"'] & \Sigma_i\overline{\Phi^i}\Phi^i \arrow[d, "(\varphi^u;\Phi^u(\Sigma_j\zeta^j))'\Phi^i"] \\
\Sigma_j\Phi^u \arrow[r, "\Phi^u(\Sigma_j\zeta^j)"'] & \Sigma_j\overline{\Phi^j}\Phi^j\Phi^u = \Sigma_j\overline{\Phi^j}\Phi^i
\end{tikzcd}
$$

In order to prove that $(\varphi^u;\Phi^u(\Sigma_j\zeta^j))'$ is indeed a theory morphism we have to show that

$$(\varphi^u;\Phi^u(\Sigma_j\zeta^j))' \ (\alpha^i_{\Sigma_i\overline{\Phi^i}}((\Sigma_i\zeta^i)(E_i))) \subseteq \alpha^j_{\Sigma_j\overline{\Phi^j}}((\Sigma_j\zeta^j)(E_j))^\bullet$$

But

$$
\begin{aligned}
& Sen^i(\Sigma_i\zeta^i);\alpha^i_{\Sigma_i\overline{\Phi^i}};Sen^0((\varphi^u;\Phi^u(\Sigma_j\zeta^j))') \\
(\text{naturality of } \alpha^i) \quad &= Sen^i(\Sigma_i\zeta^i);Sen^i((\varphi^u;\Phi^u(\Sigma_j\zeta^j))'\Phi^i);\alpha^i_{\Sigma_j\overline{\Phi^j}} \\
(\text{functoriality of } Sen^i) \quad &= Sen^i(\Sigma_i\zeta^i;(\varphi^u;\Phi^u(\Sigma_j\zeta^j))'\Phi^i);\alpha^i_{\Sigma_j\overline{\Phi^j}} \\
(\text{adjoint pair } \overline{\Phi^i},\Phi^i) \quad &= Sen^i(\varphi^u;\Phi^u(\Sigma_j\zeta^j));\alpha^i_{\Sigma_j\overline{\Phi^j}} \\
(\text{functoriality of } Sen^i) \quad &= Sen^i(\varphi^u);Sen^i(\Phi^u(\Sigma_j\zeta^j));\alpha^i_{\Sigma_j\overline{\Phi^j}} \\
(\text{the syntax part of the co-cone property} \\
\text{of institution embeddings}) \quad &= Sen^i(\varphi^u);Sen^i(\Phi^u(\Sigma_j\zeta^j));\alpha^u_{\Sigma_j\overline{\Phi^j}\Phi^j};\alpha^j_{\Sigma_j\overline{\Phi^j}} \\
(\text{naturality of } \alpha^u) \quad &= Sen^i(\varphi^u);\alpha^u_{\Sigma_j};Sen^j(\Sigma_j\zeta^j);\alpha^j_{\Sigma_j\overline{\Phi^j}}
\end{aligned}
$$

Then

$$
\begin{aligned}
(\varphi^u;\Phi^u(\Sigma_j\zeta^j))' \ (\alpha^i_{\Sigma_i\overline{\Phi^i}}((\Sigma_i\zeta^i)(E_i))) \quad &= \alpha^j_{\Sigma_j\overline{\Phi^j}}(Sen^j(\Sigma_j\zeta^j)(\alpha^u_{\Sigma_j}(Sen^i(\varphi^u)(E_i)))) \\
(\text{since } \varphi^u \text{ is extra theory morphism}) \quad &\subseteq \alpha^j_{\Sigma_j\overline{\Phi^j}}(Sen^j(\Sigma_j\zeta^j)(E_j))
\end{aligned}
$$

Finally, the functoriality of mapping the J-diagram of extra theory morphisms into a J-diagram of (intra) theory morphisms can be easily checked by simple diagram pasting.

2. By Proposition 10, φ^i are extra theory morphisms $T_i \to T_0$, for $i \in |J|$. We still have to prove the co-cone property, i.e., that $\varphi^i = \varphi^u;\varphi^j$ as extra theory morphisms, for each $u \in J(i,j)$. This means $\varphi^i = \varphi^u;\varphi^j\Phi^u$:

$$\begin{array}{rcl}
& & \varphi^i \\
(\text{universal property of } \Sigma_i\zeta^i) & = & \Sigma_i\zeta^i;(\varphi^i)'\Phi^i \\
(\text{co-cone property of } (\varphi^i)'_{i\in|J|}) & = & \Sigma_i\zeta^i;(\varphi^u;\Phi^u(\Sigma_j\zeta^j))'\Phi^i;(\varphi^j)'\Phi^i \\
(\text{universal property of } \Sigma_i\zeta^i) & = & \varphi^u;\Sigma_j\zeta^j\Phi^u;(\varphi^j)'\Phi^i \\
& = & \varphi^u;\Sigma_j\zeta^j\Phi^u;(\varphi^j)'\Phi^j\Phi^u \\
(\text{universal property of } \Sigma_j\zeta^j) & = & \varphi^u;\varphi^j\Phi^u
\end{array}$$

3. For each $i \in |J|$, let $(\theta^i)' \colon \Sigma_i\overline{\Phi^i} \to \Sigma_1\Phi^0$ be the free extension of $\theta^i \colon \Sigma_i \to \Sigma_1\Phi^0\Phi^i$. The $(\theta^i)'_{i\in|J|}$ form a co-cone over the J-diagram $sign^{\mathfrak{I}^0}(\mathbf{T}_i)_{i\in J}$ (the proof is similar to the proof of part 2. of this theorem).

Because the forgetful functor $sign^{\mathfrak{I}^0} \colon \mathbb{T}h(\mathfrak{I}^0) \to \mathbb{S}ign^0$ creates[7] co-limits (see [17]), we have that $(\varphi^i)'_{i\in|J|}$ is a co-limit for $sign^{\mathfrak{I}^0}(\mathbf{T}_i)_{i\in J}$. The $\delta \colon \Sigma_0 \to \Sigma_1\Phi^0$ should be the *unique* signature morphism such that $(\varphi^i)';\delta = (\theta^i)'$ for all $i \in |J|$.

The rest of the proof shows that δ is indeed a extra theory morphism $(\Sigma_0, E_0) \to (\Sigma_1, E_1)$, which means $\alpha^0_{\Sigma_1}(\delta(E_0)) \subseteq E_1$.

By the fundamental result on theory co-limits (see [17]), we know that E_0 is the closure of

$$\bigcup_{i\in|J|}(\varphi^i)'(\alpha^i_{\Sigma_i\overline{\Phi^i}}((\Sigma_i\zeta^i)(E_i)))$$

Therefore it is enough if we proved that for all $i \in |J|$,

$$\alpha^0_{\Sigma_1}(\delta((\alpha^i_{\Sigma_i\overline{\Phi^i}}((\Sigma_i\zeta^i)(E_i))))) \subseteq E_1$$

But

$$\begin{array}{rcl}
\multicolumn{3}{l}{Sen^i(\Sigma_i\zeta^i);\alpha^i_{\Sigma_i\overline{\Phi^i}};Sen^0(\varphi^i)';Sen^0(\delta);\alpha^0_{\Sigma_1}} \\
(\text{naturality of } \alpha^i) & = & Sen^i(\Sigma_i\zeta^i);Sen^i((\varphi^i)'\Phi^i);\alpha^i_{\Sigma_0};Sen^0(\delta);\alpha^0_{\Sigma_1} \\
(\text{universal property of } \Sigma_i\zeta^i & & \\
\text{and functoriality of } Sen^i) & = & Sen^i(\varphi^i);\alpha^i_{\Sigma_0};Sen^0(\delta);\alpha^0_{\Sigma_1} \\
(\text{naturality of } \alpha^i) & = & Sen^i(\varphi^i);Sen^i(\delta\Phi^i);\alpha^i_{\Sigma_1\Phi^0};\alpha^0_{\Sigma_1} \\
(\text{functoriality of } Sen^i) & = & Sen^i(\theta^i);\alpha^i_{\Sigma_1\Phi^0};\alpha^0_{\Sigma_1}
\end{array}$$

Then the conclusion follows because $\alpha^0_{\Sigma_1}(\alpha^i_{\Sigma_1\Phi^0}(\theta^i(E_i))) \subseteq E_1$ since $\theta^i \colon (\Sigma_i, E_i) \to (\Sigma_1, E_1)$ is a extra theory morphism. $\boxed{17}$

Corollary 18: A diagram of extra theory morphisms has a co-limit whenever the co-limit co-cone of the underlying diagram of institution morphisms consists of institution embeddings. $\boxed{18}$

Corollary 19: *Consider a partially ordered set* $(\mathrm{INST}, \sqsubseteq)$ *of institutions, where the ordering is given by institution embeddings. If* $(\mathrm{INST}, \sqsubseteq)$ *has finite least upper bounds and the category of signatures of each institution in* INST *has finite co-limits, then the category* $\mathbb{T}h(\mathrm{INST}, \sqsubseteq)$ *of the extra theory morphisms corresponding to* $(\mathrm{INST}, \sqsubseteq)$ *has finite co-limits.* $\boxed{19}$

This corollary applies to the case of the CafeOBJ cube presented in the Appendix.

[7] Using [27] terminology; this means it lifts them uniquely by terminology of [1].

3.4 Exactness

In this section we study the amalgamation property for consistent models within the general framework of extra theory morphisms.

Consider the following pushout of extra theory morphisms in the sense of Section 3.3

$$\begin{tikzcd} T \arrow[r, "\varphi^{u1}"] \arrow[d, "\varphi^{u2}"'] & T_1 \arrow[d, "\varphi^1"] \\ T_2 \arrow[r, "\varphi^2"'] & T_0 \end{tikzcd} \qquad \begin{tikzcd} \Im \arrow[r, "(\Phi^{u1},\alpha^{u1},\beta^{u1})"] \arrow[d, "(\Phi^{u2},\alpha^{u2},\beta^{u2})"'] & \Im_1 \arrow[d, "(\Phi^1,\alpha^1,\beta^1)"] \\ \Im_2 \arrow[r, "(\Phi^2,\alpha^2,\beta^2)"'] & \Im_0 \end{tikzcd}$$

where (by consistency with the notations of Theorem 17) $(\Phi^{ui},\alpha^{ui},\beta^{ui})\colon \Im \to \Im_i$ are the institution morphisms underlying φ^{ui} and $(\Phi^i,\alpha^i,\beta^i)\colon \Im_i \to \Im_0$ are the institution embeddings underlying φ^i, for $i \in \overline{1,2}$.

Then, exactness for extra theory morphisms means that the corresponding diagram of model reducts

$$\begin{tikzcd} \mathrm{MOD}(T) & \mathrm{MOD}^1(T_1) \arrow[l, "-\restriction_{\varphi^{u1}}"'] \\ \mathrm{MOD}^2(T_2) \arrow[u, "-\restriction_{\varphi^{u2}}"] & \mathrm{MOD}^0(T_0) \arrow[l, "-\restriction_{\varphi^2}"] \arrow[u, "-\restriction_{\varphi^1}"'] \end{tikzcd}$$

is a pullback. Unfortunately, such a result is not possible in the general case even when the institution embeddings involved have good properties. Very informally, this is basically due to the possibility that $\Im_1$ and $\Im_2$ share some semantic structure which does not exist in $\Im$.

Fortunately, some special cases of exactness for extra theory morphisms are enough to explain most practical situations. An important special case is given by the pushout between an intra and an extra theory morphism.

Theorem 20: Consider an institution embedding $(\Phi,\alpha,\beta)\colon \Im \to \Im_1$ and let $\varphi^{u2}\colon T \to T_2$ be a intra theory morphism in $\Im$, and $\varphi^{u1}\colon T \to T_1$ be a extra theory morphism with (Φ,α,β) the underlying institution morphism. If $\Im_1$ is (weakly) semi-exact and (Φ,α,β) is (weakly) additive and either of the following holds:

- (Φ,α,β) is strong, or
- $\Im$ is (weakly) semi-exact and Φ is surjective on objects and full

then the corresponding diagram of model reducts

$$\begin{tikzcd} \mathrm{MOD}(T) & \mathrm{MOD}^1(T_1) \arrow[l, "-\restriction_{\varphi^{u1}}"'] \\ \mathrm{MOD}(T_2) \arrow[u, "-\restriction_{\varphi^{u2}}"] & \mathrm{MOD}^1(T_0) \arrow[l, "-\restriction_{\varphi^2}"] \arrow[u, "-\restriction_{\varphi^1}"'] \end{tikzcd}$$

is a (weak) pullback.

Proof: Using similar notations to those of Theorem 17, by explicitating the model reduct functors (Proposition 11), we get that the square of model reducts to be proved (weak) pullback can be decomposed into the following tower consisting of 3 commutative squares.

$$\begin{tikzcd}
\mathrm{Mod}(\Sigma, E) & & \mathrm{Mod}(\Sigma_2, E_2) \arrow[ll, "\mathrm{Mod}(\varphi^{u2})"'] \\
\mathrm{Mod}(\Sigma\overline{\Phi}\Phi, (\Sigma\zeta)(E)) \arrow[u, "\mathrm{Mod}(\Sigma\zeta)"] & & \mathrm{Mod}(\Sigma_2\overline{\Phi}\Phi, (\Sigma_2\zeta)(E_2)) \arrow[ll, "\mathrm{Mod}(\varphi^{u2}\overline{\Phi}\Phi)"] \arrow[u, "\mathrm{Mod}(\Sigma_2\zeta)"'] \\
\mathrm{Mod}^1(\Sigma\overline{\Phi}, \alpha_{\Sigma\overline{\Phi}}((\Sigma\zeta)(E))) \arrow[u, "\beta_{\Sigma\overline{\Phi}}"] & & \mathrm{Mod}^1(\Sigma_2\overline{\Phi}, \alpha_{\Sigma_2\overline{\Phi}}((\Sigma_2\zeta)(E_2))) \arrow[ll, "\mathrm{Mod}^1(\varphi^{u2}\overline{\Phi})"] \arrow[u, "\beta_{\Sigma_2\overline{\Phi}}"'] \\
\mathrm{Mod}^1(\Sigma_1, E_1) \arrow[u, "\mathrm{Mod}^1((\varphi^{u1})')"] & & \mathrm{Mod}^1(\Sigma_0, E_0) \arrow[ll, "\mathrm{Mod}^1((\varphi^1)')"] \arrow[u, "\mathrm{Mod}^1(\varphi^{2\prime})"']
\end{tikzcd}$$

The top square commutes by the naturality of ζ, the middle one by the naturality of β extended from signatures to theories by using the Satisfaction Condition, and the bottom one by the construction of T_0 as a theory pushout in $\mathfrak{I}$ (Theorem 17).

Then the bottom square is a (weak) pullback because $\mathfrak{I}_1$ is (weakly) semi-exact. The middle square is also a (weak) pullback because (Φ, α, β) is (weakly) additive (extended to theories by using the Satisfaction Condition). The top square collapses if (Φ, α, β) is strong and is (weak) pullback when $\mathfrak{I}$ is (weakly) semi-exact and Φ is surjective on objects and full. The latter holds because Φ being surjective on objects and full implies that the underlying square of signature morphisms in $\mathfrak{I}$ is a weak pushout (by routine manipulation of the hypotheses) and this lifts to the corresponding theories.

Finally, the big square is a (weak) pullback as a composite of (weak) pullback squares. $\boxed{20}$

An important open question of this research is finding other relevant sufficient conditions for weak exactness in the case of extra theory morphisms.

3.5 Inclusion systems

As mentioned above, inclusion systems where first introduced by [15] for the institution-independent study of structuring specifications. They provide the underlying mathematical concept for module imports, which are the most fundamental structuring construct. In this paper we use the *weak inclusion systems* of [6], which constitute a improvement of the original definition of inclusion systems of [15].

Definition 21: $\langle \mathcal{I}, \mathcal{E} \rangle$ is a **weak inclusion system** for a category $\mathbb{C}$ if $\mathcal{I}$ and $\mathcal{E}$ are two sub-categories with $|\mathcal{I}| = |\mathcal{E}| = |\mathbb{C}|$ such that

1. $\mathcal{I}$ is a partial order, and
2. every arrow f in $\mathbb{C}$ can be factored uniquely as $f = e;i$ with $e \in \mathcal{E}$ and $i \in \mathcal{I}$.

The arrows of $\mathcal{I}$ are called **inclusions**, and the arrows of $\mathcal{E}$ are called **surjections**.[8]. The domain (source) of the inclusion i in the factorization of f is called called the **image of** f and denoted as $\mathrm{Im}(f)$. $\boxed{21}$

For the fundamental properties of weak inclusion systems and techniques to construct them consult [6].

We need the following technical definition:

Definition 22: Let $\mathbb{C}$ and $\mathbb{C}'$ be two categories with weak inclusion systems $\langle \mathcal{I}, \mathcal{E} \rangle$ and $\langle \mathcal{I}', \mathcal{E}' \rangle$ respectively. Then a functor $\mathcal{U}\colon \mathbb{C} \to \mathbb{C}'$ **lifts inclusions uniquely** iff for any inclusion $\iota'\colon A' \hookrightarrow B\mathcal{U}$ in $\mathcal{I}'$ with $B \in |\mathbb{C}|$, there exists a unique inclusion $\iota \in \mathcal{I}$ such that $\iota\mathcal{U} = \iota'$. $\boxed{22}$

[8] Surjections of some weak inclusion systems need not necessarily be surjective in the ordinary sense.

Theorem 23: Consider a category of institutions with a weak inclusion system $\langle \mathcal{I}^{\mathrm{INST}}, \mathcal{E}^{\mathrm{INST}} \rangle$ such that each of institutions involved $\mathfrak{I} = (\mathbb{S}ign, \mathrm{MOD}, Sen, \models)$ has a weak inclusion system $\langle \mathcal{I}^{\mathfrak{I}}, \mathcal{E}^{\mathfrak{I}} \rangle$ for its category of signatures. If

- Φ preserves inclusions for each $(\Phi, \alpha, \beta) \in \mathcal{I}^{\mathrm{INST}}$, and
- Φ preserves both inclusions and surjections and lifts inclusions uniquely for each $(\Phi, \alpha, \beta) \in \mathcal{E}^{\mathrm{INST}}$,

then the corresponding category of extra theory morphisms has an inclusion system where $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ is

- *inclusion* iff both the underlying institution morphism $(\Phi, \alpha, \beta)\colon \mathfrak{I} \to \mathfrak{I}'$ and the signature morphism $\varphi\colon \Sigma \to \Sigma'\Phi$ are inclusions,
- *surjection* iff both the underlying institution morphism $(\Phi, \alpha, \beta)\colon \mathfrak{I} \to \mathfrak{I}'$ and the signature morphism $\varphi\colon \Sigma \to \Sigma'\Phi$ are surjections, and if $\alpha_{\Sigma'}(\varphi(E))^{\bullet} = E'$.

Proof: Let us denote by $\mathcal{I}^{\mathbb{T}h}$ the subcategory of inclusion extra theory morphisms and by $\mathcal{E}^{\mathbb{T}h}$ the subcategory of surjection extra theory morphisms.

Consider two extra theory morphisms $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ and $\varphi\colon (\Sigma', E') \to (\Sigma'', E'')$ with $(\Phi, \alpha, \beta)\colon \mathfrak{I} \to \mathfrak{I}'$ and $(\Phi', \alpha', \beta')\colon \mathfrak{I}' \to \mathfrak{I}''$ as underlying institution morphisms. If both of them are inclusions in $\mathcal{I}^{\mathbb{T}h}$, then (Φ, α, β) and (Φ', α', β') are inclusions in $\mathcal{I}^{\mathrm{INST}}$, hence their composition is an inclusion in $\mathcal{I}^{\mathrm{INST}}$ too. Also, $\varphi; \varphi'\Phi$ is an inclusion as a composite of two inclusions in $\mathcal{I}^{\mathfrak{I}}$ where the latter is an inclusion because Φ preserves inclusions. A similar argument holds for compositions of surjections; however in the case of surjections we also have to check the closure condition. This follows by routine calculation.

Now, let $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ be an arbitrary extra theory morphism with $(\Phi, \alpha, \beta)\colon \mathfrak{I} \to \mathfrak{I}'$ as its underlying institution morphism. Then (Φ, α, β) factors uniquely as

$$\begin{tikzcd}
\mathfrak{I} \arrow[rr, "{(\Phi,\alpha,\beta)}"] \arrow[dr, "{(\Phi^{\mathrm{e}},\alpha^{\mathrm{e}},\beta^{\mathrm{e}})}"'] & & \mathfrak{I}' \\
& \mathfrak{I}'' \arrow[ur, "{(\Phi^{\mathrm{i}},\alpha^{\mathrm{i}},\beta^{\mathrm{i}})}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma \arrow[rr, "\varphi"] \arrow[dr, "\varphi^{\mathrm{e}}"'] & & \Sigma'\Phi^{\mathrm{i}}\Phi^{\mathrm{e}} \\
& \Sigma^{1} \arrow[ur, "\varphi^{1}"'] &
\end{tikzcd}$$

where $(\Phi^{\mathrm{e}}, \alpha^{\mathrm{e}}, \beta^{\mathrm{e}}) \in \mathcal{E}^{\mathrm{INST}}$ and $(\Phi^{\mathrm{i}}, \alpha^{\mathrm{i}}, \beta^{\mathrm{i}}) \in \mathcal{I}^{\mathrm{INST}}$ and φ factors uniquely through the weak inclusion system $\langle \mathcal{I}^{\mathfrak{I}}, \mathcal{E}^{\mathfrak{I}} \rangle$. Since Φ^{e} lifts inclusions uniquely there exists an unique inclusion $\varphi^{\mathrm{i}}\colon \Sigma'' \hookrightarrow \Sigma'\Phi^{\mathrm{i}}$ such that $\varphi^{\mathrm{i}}\Phi^{\mathrm{e}} = \varphi^{1}$. We then define $E'' = \alpha^{\mathrm{e}}_{\Sigma''}(\varphi^{\mathrm{e}}(E))^{\bullet}$. Therefore, $\varphi^{\mathrm{e}}\colon (\Sigma, E) \to (\Sigma'', E'')$ is a surjection extra theory morphism and $\varphi^{\mathrm{i}}\colon (\Sigma'', E'') \to (\Sigma', E')$ is an inclusion extra theory morphism.

Finally, the uniqueness of this factorization follows stepwise from the uniqueness of the factorization of the underlying institution morphism, then from the uniqueness of the factorization through the inclusion system of $\mathfrak{I}$ (by using the preservation of inclusions by the Φ^{e}), then from the uniqueness of the lifting to $\mathcal{I}^{\mathfrak{I}''}$, and finally from the closure condition on sentences. $\boxed{23}$

Practical applications use mostly the following much simpler Corollary:

Corollary 24: Consider a partial ordered set of institutions and institution morphisms such that each of institutions involved $\mathfrak{I} = (\mathbb{S}ign, \mathrm{MOD}, Sen, \models)$ has a weak inclusion system $\langle \mathcal{I}^{\mathfrak{I}}, \mathcal{E}^{\mathfrak{I}} \rangle$ for its category of signatures with Φ preserving inclusions for each institution morphism (Φ, α, β). Then the corresponding category of extra theory morphisms has an inclusion system where $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ is

- *inclusion* iff the signature morphism $\varphi\colon \Sigma \to \Sigma'\Phi$ is an inclusion in $\mathcal{I}^{\mathfrak{I}}$,
- *surjection* iff the underlying institution morphism is identity and it is a surjection in $\mathcal{E}^{\mathfrak{I}}$

Proof: By considering the inclusion system of the partially ordered set of institutions with all institution morphisms as inclusions and identities as surjections. $\boxed{24}$

4 Logical Semantics for Multi-Paradigm Languages

In this section we outline a general logical semantics for multi-paradigm specification/programming languages; this semantics is based on the concept of extra theory morphism. We assume the following general framework and principles:

1. There is a lattice (the partial order being denoted as $\sqsubseteq$) of institution embeddings in which all basic constructs/features of the language can be rigorously explained.
2. Each institution corresponds to a language paradigm, institution embeddings corresponding to paradigm embeddings.
3. For a given language paradigm, the basic specifications are assimilated to the theories generated in the corresponding institution.

This intimate relationship between the language and its "underlying logic" (in this case given by the lattice of institution embeddings) was first conceptualized by Goguen and Meseguer [22] under the name of *logical programming*. As mentioned in the Introduction, such logical languages include most of the OBJ family of languages. In the Appendix we illustrate our logical semantics with the example of CafeOBJ [14], a modern successor of OBJ.

4.1 Basic Specifications

At the level of basic specifications (the so-called "programming in-the small"), we have two kinds of semantics, *tight*, and *loose*. Given a basic specification T (regarded as a theory in an institution $\mathfrak{I}$), its **tight denotation** is the initial model 0_T of $\mathrm{MOD}(T)$, and its **loose denotation** is given by all models in $\mathrm{MOD}(T)$. The notation for the denotation of a specification T is $[\![T]\!]$. To resume

$$[\![T]\!] = \begin{cases} 0_T & \text{if initial semantics} \\ \mathrm{MOD}(T) & \text{if loose semantics}. \end{cases}$$

4.2 Structured Specifications

This is the level of the logical semantics where most of the results on extra theory morphisms apply. We extend the basic concepts of the OBJ structuring mechanism (or module system) which are inherited from earlier work on Clear [4] and further developed at the institutional level in [15] to the more refined situation of lattice of institution embeddings.

The concept of *model expansion* is dual to model reducts, and plays a crucial rôle for defining the denotations of structured specifications:

Definition 25: Given an extra theory morphism $\varphi\colon T \to T'$, and a model M of T, an **expansion of** M **along** φ is a model M' of T' satisfying the following properties:

- $M'\restriction_\varphi = M$ iff the expansion is **protecting**,
- there is an *injective*[9] model homomorphism $M \hookrightarrow M'\restriction_\varphi$ iff the expansion is **extending**,
- there is an arbitrary model homomorphism $M \to M'\restriction_\varphi$ iff the expansion is **using**, and
- M' is the free over M with respect to φ (see Definition 12) iff the expansion is **free**.

$\boxed{25}$

The general structuring mechanism is constituted by *module expressions*, which are iterations of several basic structuring operations, such as imports, parameters, instantiation of parameters by views, translations, etc. In this section we discuss the most important ones: imports and parameterization.

[9] Under a suitable concept of "injectivity".

4.2.1 Imports

Module imports constitute the primitive concept underlying any module interconnection systems, here is it's mathematical definition:

Definition 26: A **module import** $T \trianglelefteq T'$ is an inclusion extra theory morphism $T \hookrightarrow T'$, where the institution $\mathfrak{I}_{T'}$ of T' embeds the institution $\mathfrak{I}_T$ of T (i.e., $\mathfrak{I}_T \sqsubseteq \mathfrak{I}_{T'}$ in the fixed lattice of institutions). $\boxed{26}$

The (weak) inclusion system underlying this definition is that introduced by Corollary 24 for the current lattice of institution embeddings.

By following the OBJ tradition, we can distinguish between 3 basic kinds of imports, **protecting**, **extending**, and **using**. At the level of the language, these should be treated just as semantic declarations which determine the denotation of the importing module from the denotation of the imported module.

Definition 27: Fix an import $T \trianglelefteq T'$. Then $\llbracket T' \rrbracket =$

- $\{M' \mid M' \models T', M'$ protecting (and free if T' is initial) expansion of a model $M \in \llbracket T \rrbracket\}$, when the importation mode is *protecting*,
- $\{M' \mid M' \models T', M'$ extending (and free if T' is initial) expansion of a model $M \in \llbracket T \rrbracket\}$, when the importation mode is *extending*, and
- $\{M' \mid M' \models T',$ (and free expansion of a model M of T if T' is initial)$\}$, when the importation mode is *using*.

$\boxed{27}$

Multiple imports are handled by a lattice structure on inclusions (see [15, 6].

Definition 28: Given two modules T and T', their **shared part** $T \cap T'$ is the *greatest lower bound* in the lattice of imports $\trianglelefteq$ and their **sum** $T + T'$ is the *lowest upper bound*. $\boxed{28}$

We can easily notice the that the institution of the sum unifies the paradigms of the institution of the components:

Fact 29: $\mathfrak{I}_{T+T'} = \mathfrak{I}_T \sqcup \mathfrak{I}_{T'}$, i.e., the lowest upper bound of $\mathfrak{I}_T$ and $\mathfrak{I}_{T'}$ in the lattice of institution embeddings. $\boxed{29}$

The following extends a basic result on multiple imports from [15] to the multi-paradigm case:

Corollary 30: Let T and T' be two modules. Then we have the following pushout-pullback square (in $\mathfrak{I}_T \sqcup \mathfrak{I}_{T'}$)

$$\begin{tikzcd} T \arrow[r, "\trianglelefteq"] & T + T' \\ T \cap T' \arrow[u, "\trianglelefteq"] \arrow[r, "\trianglelefteq"'] & T' \arrow[u, "\trianglelefteq"'] \end{tikzcd}$$

where $T \cap T'$ is the shared part (i.e., the intersection) of T and T'. $\boxed{30}$

4.2.2 Parameterization

Parameterized specification/programming is a very important feature of all modern declarative languages. The mathematical definition of parameterized modules is based on the following concept of *injection*:

Definition 31: Given a (weak) inclusion system, an **injection** is the composite between an inclusion and an isomorphism. $\boxed{31}$

Definition 32: A **parameterized specification (module)** $T(X :: P)$ is an *injection* $P \xrightarrow{X} T$. The institution of the parameter P is embedded in the institution of the body T (i.e., $\mathfrak{I}_P \sqsubseteq \mathfrak{I}_T$).

A **view** is just an extra theory morphism. $\boxed{32}$

We distinguish two basic approaches on parameters: a "shared" and a "non-shared' one. In the "non-shared" approach, the multiple parameters are mutually disjoint (i.e., $\mathrm{Im}(X) \cap \mathrm{Im}(X') = \emptyset$ for X and X' two different parameters) and they are also disjoint from any module imports $T_0 \trianglelefteq T$ (i.e., $\mathrm{Im}(X) \cap T_0 = \emptyset$). In the "shared" approach this principle is relaxed to being disjoint *outside common imports*, i.e., $\mathrm{Im}(X) \cap \mathrm{Im}(X') = \sum_{T_1 \trianglelefteq X} T_1 \cap \sum_{T_1 \trianglelefteq X'} T_1$ for X and X' two different parameters and $\mathrm{Im}(X) \cap T_0 = \sum_{T_1 \trianglelefteq X} \cap T_0$ for all $T_0 \trianglelefteq T$. The "non-shared" approach has the potentiality of a much more powerful module system, while the "shared" approach seems to be more convenient to implement. The CafeOBJ definition contains both of them, for details on "non-shared" vs. "shared" parameterization and for a more detailed presentation of a module system based on this theory, see [14].

Definition 33: Let $T(X :: P)$ be a parameterized module and $v \colon P \to P'$ be a view. Then the instantiation $T(v)$ is given by the following pushout in the sense of Theorem 17

$$\begin{tikzcd}
P \arrow[r, "X"] \arrow[d, "v"'] & T \arrow[d, "v'"] \\
P' \arrow[r, "X(v)"'] & T(v)
\end{tikzcd}$$

in the "non-shared" approach and by the following co-limit in the sense of Theorem 17

$$\begin{tikzcd}
P' \cap T \arrow[drr, "\trianglelefteq"] \arrow[ddr, "\trianglelefteq"'] & & \\
& P \arrow[r, "X"'] \arrow[d, "v"'] & T \arrow[d, "v'"] \\
& P' \arrow[r, "\trianglelefteq"'] & T(v)
\end{tikzcd}$$

in the "shared" approach. In both cases the embedding institution is $\mathfrak{I}_T \sqcup \mathfrak{I}_{P'}$. $\boxed{33}$

5 Conclusions and Future Research

We have defined a more general concept of theory morphism mapping theories across institution morphisms. This generalizes the ordinary concept of theory morphism which is confined to single institutions. We have lifted the basic concepts related to theory morphisms from the ordinary case to the "extra" case, and we have investigated the basic properties of extra theory morphisms. We have proved the following results:

- Existence of model reducts for extra theory morphisms (equivalent to a Satisfaction Condition for extra theory morphisms),

- Sufficient conditions for free constructions along extra theory morphisms,
- Construction of theory co-limits,
- Exactness (model amalgamation) properties, and
- Inclusion systems for extra theory morphisms.

We have also sketched a generic logical semantics for multi-paradigm languages which is based on extra theory morphisms.

Future research directions include the full development of a general logical semantics based on extra theory morphisms (including a corresponding "module algebra"), and further investigations of sufficient conditions for exactness properties.

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A CafeOBJ

As previously mentioned, CafeOBJ [14], currently under development in Japan,[10] is a modern successor of the famous algebraic language OBJ. CafeOBJ adds new basic paradigms such as behavioural concurrent specification [20] and rewriting logic [5]. The following table shows the correspondence between specification/programming paradigms and institutions as they appear in the actual design of CafeOBJ, also pointing to some basic references.

ABBREVIATION	LOGIC	SPEC/PGM PARADIGM	BASIC REF.
MSA	many sorted algebra	algebraic specification	[16]
OSA	order sorted algebra	algebraic specification with subtypes	[16, 23, 18]
HSA	hidden sorted algebra	behavioural concurrent specification	[**?**]
HOSA	hidden order sorted algebra	behavioural specification with subtypes	[**?**, 3]
RWL	rewriting logic	rewriting logic specification	[28]
OSRWL	order sorted rewriting logic	rewriting logic specification with subtypes	
HSRWL	hidden sorted rewriting logic	behavioural rewriting logic specification	[12, 9]
HOSRWL	hidden order sorted rewriting logic	behavioural rewriting logic specification with subtypes	

An approximation of the lattice of the institution embeddings involved is given by the following CafeOBJ cube:

Other "dimensions" might be added to this cube, most notably the *constraint logic* [7, 10] which give elegant semantics to pre-defined data types and to libraries.

All institution morphisms of the CafeOBJ cube are strong, persistent, and additive embeddings. The symmetry of the CafeOBJ cube means it is a lattice, therefore all basic hypotheses of the logical semantics of Section 4 are fulfilled. HOSRWL embeds all other institutions, hence it represents the flattening of the cube; below we briefly present it. However, it is important to consider the CafeOBJ cube in its entirety rather than HOSRWL alone since some subtle information on the relationship between the component features is lost in this flattening. Such flattening works well only when all institution embeddings involved have the components of the model translations (β) as equivalence of categories[11], but in the case of the

[10] Project supported on a large scale by the Japanese Government through its Information Promotion Agency.

[11] Called just "institution embeddings" in [29].

CafeOBJ cube this property does not hold along the RWL dimension because of forgetting the transitions from the RWL models.

Hidden Order Sorted Rewriting Logic

We devote this appendix to the (rather informal) presentation in some detail of HOSRWL (first introduced in [12] in the many sorted version) which embeds all CafeOBJ cube institutions. However, the deep understanding of HOSRWL requires further reading on its main components ([28] for RWL and [**?**, 20] for HSA) as well as their integration [12]. We assume familiarity with basic many sorted algebra which constitutes the underlying level of all algebraic specification developments (relevant background can be found in [16, 24, 30]), but also with order sorted algebra [23, 18].

Signatures

A **hidden signature** is a tuple $(H,V,\leq,\Psi,\Sigma,\Sigma^{\mathrm{b}})$, where

- $(H,\leq)$ is a partially ordered set of **hidden sorts**,
- $(V,\leq)$ is a partially ordered set of **visible sorts**,
- $(H,\leq)$ and $(V,\leq)$ are disjoint,
- Ψ is an $(V,\leq)$-order-sorted (o.s., for short) signature,
- Σ is an $(H\cup V,\leq)$-o.s. signature,

(S1) each $\sigma \in \Sigma_{w,s}$ with $w \in V^*$ and $s \in V$ lies in $\Psi_{w,s}$,

- $\Sigma^{\mathrm{b}} \subseteq \Sigma$ is a marked sub-signature of **behavioural operations** such that $\Sigma^{\mathrm{b}} \cap \Psi = \emptyset$, and

(S2) each $\sigma \in \Sigma_{w,s}$ has *exactly* one element of H in w.

The operations in Σ^{b} have object-oriented meaning, $\sigma \in \Sigma^{\mathrm{b}}_{w,s}$ is **method** if s is hidden and **attribute** if s is visible. Condition (S1) is a data encapsulation condition, and (S2) says that methods and attributes act on (states of) single objects.

A **hidden rewrite signature** is given by $(H,V,\leq,\Psi,\Sigma,\Sigma^{\mathrm{b}},E)$ where $(H,V,\leq,\Psi,\Sigma,\Sigma^{\mathrm{b}})$ is a hidden o.s. signature and E is a collection of Σ-equations. A **hidden sorted rewrite signature morphism** $\phi\colon (H,V,\leq,\Psi,\Sigma,\Sigma^{\mathrm{b}},E) \to (H',V',\leq,\Psi',\Sigma',\Sigma' b,E')$ is an o.s. signature morphism $(H\cup V,\leq,\Sigma) \to (H'\cup V',\leq,\Sigma')$ such that

(M1) $\phi(\Psi) \subseteq \Psi'$,

(M2) $\phi(H) \subseteq H'$ and $\phi(\Sigma^{\mathrm{b}}) \subseteq \Sigma' b$,

(M3) if $\sigma' \in \Sigma'^{\mathrm{b}}_{w',s'}$ and some sort in w' lies in H', then $\sigma' = \phi(\sigma)$ for some $\sigma \in \Sigma^{\mathrm{b}}$,

(M4) if $\phi(h) < \phi(h')$ for any hidden sorts $h,h' \in H$, then $h < h'$, and

(M5) $E' \models_{\Sigma'} \phi(E)$.

The first two conditions say that hidden sorted signature morphisms preserve visibility and invisibility for both sorts and operations, the third[12] and fourth conditions express the encapsulation of classes and subclasses (in the sense that no new methods or attributes can be defined on an imported class), while the fifth expresses the encapsulation of structural axioms.

[12] Without (M3) the Satisfaction Condition fails, for more details on the logical and computational relevance of (M3) see [**?**].

Sentences

Given a signature $(H,\leq,\Sigma,E)$, a sentence is either a (possibly conditional) **equation** (modulo E) or else a (possibly conditional) **transition** (modulo E). Since equations are very traditional to algebraic specification, we concentrate here on transitions. A conditional transition is written as

$$(\forall X)\ [t] \to [t'] \textbf{ if } [u_1] \to [v_1] \ldots [u_k] \to [v_k]$$

where t,t',u_i,v_i are Σ-terms with variables X and modulo the equations in E (i.e., equivalence classes of Σ-terms modulo the congruence determined by E). The left-hand side of **if** is the head of the transition and the right-hand side is the condition of the transition.

Given a signature morphism $\phi\colon (H,V,\leq,\Psi,\Sigma,\Sigma^{\mathrm{b}},E) \to (H',V',\leq,\Psi',\Sigma',\Sigma' b,E')$ the translation of sentences is defined by the translation of Σ-terms (modulo E) to Σ'-terms modulo E' along ϕ by replacing all symbols in Σ-terms with the corresponding symbols for Σ'. Condition (M5) enforces the correctness of this definition. For a full rigorous treatment of this issue the reader is advised to consult [7, 11].

Models

Given an algebraic theory (Σ,E), a **rewrite model** for (Σ,E) is given by the interpretation of the algebraic theory into $\mathbb{C}at$. More concretely, a model M interprets each sort s as a category M_s, each subsort relation $s<s'$ as sub-category relation $M_s \subseteq M_{s'}$, and each operation $\sigma \in \Sigma_{w,s}$ as a functor $\sigma_M\colon M_w \to M_s$, where M_w stands for $M_{s_1} \times \ldots \times M_{s_n}$ for $w = s_1 \ldots s_n$. Each Σ-term $t\colon w \to s$ gets a functor $t_M\colon M_w \to M_s$ by evaluating it for each assignment of the variables occurring in t with arrows from the corresponding carriers of M. The satisfaction of an equation $t = t'$ by M is given by $t_M = t'_M$;[13] in particular all structural equations should be satisfied by M. A model morphism is a family of functors indexed by the sorts commuting with the interpretations of the operations in Σ.

This algebra "enriched" over $\mathbb{C}at$ is a special case of *category-based equational logic* (see [7, 8, 19]) when letting the category $\mathbb{A}$ of models be the interpretations of Σ into $\mathbb{C}at$ as abovely described, the category $\mathbb{X}$ of domains to be the category of many sorted sets, and the forgetful functor $\mathcal{U}\colon \mathbb{A} \to \mathbb{X}$ forgetting the interpretations of the operations and the composition between the arrows, i.e., mapping each category to its set of arrows. This enables the use of the machinery of category-based equational logic as a technical aid to the model theory of RWL.

Hidden sorted models are just ordinary models (either algebras or rewrite models).

Satisfaction

Let $(H,\leq,\Sigma,E)$ be a hidden sorted signature, $[\rho]$ be a sentence,[14] and M be a model for this signature. Satisfaction in RWL of ordinary equations was explained in the paragraph on models, so we concentrate on the satisfaction of transitions.

The satisfaction of a transition $(\forall X)\ [t] \to [t'] \textbf{ if } [u_1] \to [v_1] \ldots [u_k] \to [v_k]$ by M has a rather sophisticated definition using the concept of *subequalizer*. Let w be the string of sorts associated to the collection of variables X. Then

$$M \models (\forall X)\ [t] \to [t'] \textbf{ if } [u_1] \to [v_1] \ldots [u_k] \to [v_k]$$

iff there exists a natural transformation $J_M;t_M \Rightarrow J_M;t'_M$ where $J_M\colon \mathit{Subeq}((u_{iM},v_{iM})_{i\in 1\ldots k}) \to M_w$ is the subequalizer functor, i.e., the functor component of the final object in the category having pairs $(\mathit{Dom}(S) \xrightarrow{S} M_w, (S;u_{iM} \overset{\alpha_i}{\Rightarrow}, S;v_{iM})_{i\in 1\ldots k})$ as objects and functors H such that $H;S' = S$ and $H\alpha' = \alpha$ as arrows.

Finally, the satisfaction in HOSRWL is **behavioural** (denoted by $\models\!\!\!\equiv$); for details see [12, 9].

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[13] This definition extends without difficulty to conditional equations.

[14] We extend the equivalence class notation from terms to sentences in the obvious way.