An Encoding of Partial Algebras as Total Algebras

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Abstract

We introduce a semantic encoding of partial algebras as total algebras through a Horn axiomatization of the existence equality relation interpreted as an algebraic operation. We show that this novel encoding enjoys several important properties that make it a good tool for the execution of partial algebraic specifications through means specific to ordinary algebraic reasoning, such as term rewriting.

Key words: Partial algebra; Encoding; Formal specifications

1. Introduction

Partial functions play an important role in computing science; this is well know. In particular, the specification power of operations that are partially defined makes partial algebra as one of the important formalisms employed by modern formal methods, a prominent example being the recent algebraic specification language CASL [1]. However, the current mathematical culture, including school mathematics that is responsible for the basic patterns of our mathematical thinking, is strongly biased towards total functions. Reasons are manifold. First of all, algebraic reasoning with total functions is much simpler. Then, the semantics of partial functions is significantly more sophisticated than that of ordinary total algebra. These above two aspects are of course interdependent. Consequently, mechanical algebraic reasoning with total functions is supported by a rather impressive variety of tools and execution engines, most of them based upon the so-called term rewriting method, while partial algebra formalism lacks such kind of computational infrastructure.

The above mentioned reasons have led to various efforts to translate, or encode, partial algebra into logics with total functions with the benefit of doing things in the translation and exporting the results back to the original partial algebra framework. Although such way of 'doing logic by translation' has a long tradition starting perhaps with the study of embeddings of classical into intuitionistic logic [11], this has received a new life with the arrival of the so-called institution theory [7] with its array of model theory oriented category theoretic concepts supporting the systematic studies of logic translations [13] at a higher mathematical level. A characteristic aspect of the institution theoretic translations is that due to their semantic nature they are rather fine grained, in the sense that they involve the deepest aspects of the logical systems. The translations of partial algebra into logics with total functions appear usually as 'simple theoroidal comorphisms' of [9]. One of the most prominent such translation [12] encodes partial operations as total operations but employs also relations for expressing the domains of definition of the partial operations. Another important one [12,14] encodes the partial operations as relation symbols. Both of them can be found also in the recent monography [5].

Here we propose a novel translation that, unlike the previous ones, has a pure algebraic nature since it does employ only operation symbols, no relation symbols. Thus any partial algebraic signature gets encoded as a set of conditional equations for a total algebraic signature. We show that the proposed translation enjoys some important properties, such as the so-called 'satisfaction condition' of institution comorphisms, and the so-called 'persistent liberality' property. These properties together with the above mentioned pure algebraic feature of the translation lead to an important proof theoretic consequence: a sound and complete calculus for the Horn fragment of partial algebra can be expressed as ordinary equational calculus with total operations symbols. This provides a simple and efficient way to execute Horn partial algebraic specifications by ordinary term rewriting. Another plus of our proposed translation is

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that it yields a rather simple and intuitive representation of partial algebras as total algebras.

Regarding the translation itself, it is inspired from the encoding of the conditions of the equations as Boolean terms as done by the OBJ language [10] and its succesor CafeOBJ [6]. This is based upon the encoding of the equality relation $=$ as a Boolean valued algebraic operation `==`, CafeOBJ extending this also to the transition relation (for preordered algebras) as `==>` and to the behavioural equivalence relation as `=b=`. Thus our proposed translation encodes the fundamental existence equality relation $\stackrel{e}{=}$ from partial algebra as an algebraic operation, denoted $\textcircled{e}$, and provides a set of Horn axioms for this operation.

Although our translation of partial algebras as total algebras is based upon institution theoretic principles, and can be presented as an institution comorphism, we keep our presentation as elementary as possible by avoiding any direct employment of institution theoretic notions. However we assume some familiarity with the basic category theoretic notions of category, functor, and initial objects. We also use the diagrammatic notation for the compositions of functions and homomorphism, i.e. if $f: A \to B$ and $g: B \to C$ then $f;g: A \to C$ denotes the composition between f and g.

2. Preliminaries

Now let us briefly review the basic notions and fix some notations about total algebras and about partial algebras,considered in their many sorted form.

Total algebra. (abbreviatted **ALG**)
An *algebraic signature* is pair (S, F) consisting of a set of sort symbols S and of a family $F = \{F_{w\to s} \mid w \in S^*, s \in S\}$ of sets of function symbols indexed by strings of sort symbols, called arities, (for the arguments) and sorts (for the results). *Algebras* A for a signature (S, F) interpret each sort symbol s as a set A_s and each function symbol $\sigma \in F_{w\to s}$ as a function $A_\sigma: A_w \to A_s$ where by A_w we denote the cartesian product $A_{s_1} \times \cdots \times A_{s_n}$ where $w = s_1 \ldots s_n$. An *algebra homomorphism* $h: A \to A'$ is an indexed family of functions $\{h_s: A_s \to A'_s\}_{s\in S}$ such that $h_s(A_\sigma(a)) = A'_\sigma(h_w(a))$ for each $\sigma \in F_{w\to s}$ and each $a \in A_w$. [1] For any signature (S, F), the (S, F)-algebra homomorphisms compose component-wise as functions, and this yields a category denoted $\mathbf{ALG}(S, F)$.

(S, F)-*terms* can be defined inductively as follows: for any $\sigma \in F_{s_1\ldots s_n\to s}$, a structure of the form $\sigma(t_1, \ldots, t_n)$ is a term of sort s whenever t_i are terms of sorts s_i, respectively. The *sentences* of the signature (S, F) are the usual first order sentences built from equational atoms of the form $t = t'$, where t and t' are terms of the same sort, by iterative application of Boolean connectives ($\wedge$, $\neg$, $\Rightarrow$, etc.) and quantifiers. For quantifiers this goes as follows: if X is a finite *set of variables for* (S, F), each variable $x \in X$ having a designated sort $\mathsf{sort}(x) \in S$, and if ρ is an $(S, F \cup X)$-sentence (where $F \cup X$ means $(F \cup X)_{w\to s} = F_{w\to s}$ when w is non-empty and $(F \cup X)_{\to s} = F_{\to s} \cup \{x \in X \mid \mathsf{sort}(x) = s\}$), then $(\forall X)\rho$ and $(\exists X)\rho$ are both (S, F)-sentences. By a *conditional equation* we mean any sentence of the form $(\forall X)H \Rightarrow C$ where H is a finite conjunction of equational atoms and C is a single equational atom. The satisfaction of (S, F)-sentences by (S, F)-algebras, denoted by $\models^{\mathbf{ALG}}_{(S,F)}$, is the usual Tarskian satisfaction defined inductively on the structure of the sentences. In more detail this means

- $A \models t = t'$ if and only if $A_t = A_{t'}$ where for any term t its *evaluation* in A, denoted by A_t, is defined inductively by the formula $A_{\sigma(t_1,\ldots,t_n)} = A_\sigma(A_{t_1}, \ldots, A_{t_n})$.
- $A \models \rho_1 \wedge \rho_2$ if and only if $A \models \rho_1$ and $A \models \rho_2$, $A \models \neg\rho$ if and only if $A \not\models \rho$, etc.
- $A \models (\forall X)\rho$ if and only if $A' \models \rho$ for any $(S, F \cup X)$-expansion A' of A. An $(S, F \cup X)$-algebra A' is an *expansion* of A when $A'_z = A_z$ for each $z \in S$ or $z \in F_{w\to s}$.

Partial algebra. (abbreviated **PA**) Here we refer to the partial algebra as used in CASL [12] which represents a slight refinement of the concept of partial algebra as defined in the standard textbook [3]. Another important partial algebra work within computing science is [15].

A *partial algebraic signature* is a tuple (S, TF, PF), where both (S, TF) and (S, PF) are algebraic signatures such that $TF_{w\to s}$ and $PF_{w\to s}$ are always disjoint. TF stands for 'total' function symbols while PF stands for 'partial' function symbols. A *partial algebra* A is just like a total algebra but interpreting the function symbols of PF as partial rather than total functions. This means that for each $\sigma \in PF_{w\to s}$ there is a subset $dom(A_\sigma) \subseteq A_w$ which is the domain of definition of A_σ, i.e. the subset of the arguments for which A_σ is defined. A *partial algebra homomorphism* $h: A \to B$ is a family of (total) functions $\{h_s: A_s \to B_s\}_{s\in S}$ indexed by the set of sorts S of the signature such that $h_w(A_\sigma(a)) = B_\sigma(h_s(a))$ for each operation $\sigma \in TF_{w\to s} \cup PF_{w\to s}$ and each string of arguments $a \in A_w$ for which $A_\sigma(a)$ is defined. (In particular this also implies that $h_s(a) \in dom(B_\sigma)$.) For any **PA** signature (S, TF, PF), the homomorphisms of partial (S, TF, PF)-algebras compose component-wise as functions, and this yields a category denoted $\mathbf{PA}(S, TF, PF)$.

The sentences for a signature are build like in the case of the total algebras from existence equality atoms $t \stackrel{e}{=} t'$ and by restricting the quantification only to sets of variables X that are total. The *quasi-existence equations* are the sentences of the form $(\forall X)H \Rightarrow C$ where H is any finite conjunction of existence equality atoms and C is a single existence equality atom. An existence equality $t \stackrel{e}{=} t'$ holds in an algebra A when both terms are defined and are equal. The terms are formed with function symbols from TF and PF, and a term t is defined in an algebra A when A_t can be evaluated, which means that assuming that $t = \sigma(t_1, \ldots, t_n)$ then t is defined in A when each t_i is defined

[1] If $w = s_1 \ldots s_n$ and $a = (a_1, \ldots, a_n)$ then by $h_w(a)$ we mean the tuple $(h_{s_1}(a_1), \ldots, h_{s_n}(a_n))$.

in A and $(A_{t_1}, \ldots, A_{t_n}) \in dom(A_\sigma)$; in this case $A_t = A_\sigma(A_{t_1}, \ldots, A_{t_n})$. The satisfaction of existence equalities by partial algebras can be extended to all sentences like for total algebras; note the role played by the assumption that the quantifications are total. The satisfaction relation(s) thus obtained may be denoted by $\models^{\mathbf{PA}}_{(S,TF,PF)}$.

Presentations. Given a signature Σ (either of total or partial algebras) a Σ-*presentation* consists of a set of Σ-sentences E. Σ-presentations are denoted by (Σ, E).

3. The encoding

The proposed encoding has three components. First there is the encoding of the **PA** signatures as **ALG** presentations. Then each **PA** sentence gets translated to an **ALG** sentence of the corresponding translated signature. Hence the translation of the sentences goes in the same direction with the translation of the signatures. The translation of the models however goes opposite, any (total) algebra of the encoding of a **PA** signature gets mapped to a partial algebra of that signature. This may be the most often used kind of logical encoding, known in institution theory as 'simple theoroidal comorphism' [9], which is a special kind of comorphism between institutions.

3.1. *The encoding of the signatures*

Each **PA** signature (S, TF, PF) gets mapped to an **ALG** presentation $((S \cup \{\mathbf{b}\}, TF \oplus PF), \Gamma_{(S,TF,PF)})$ where
- $(TF \oplus PF)_{w \to s} = TF_{w \to s} \cup PF_{w \to s}$ when $s \neq \mathbf{b}$,
- $(TF \oplus PF)_{ss \to \mathbf{b}} = \{\textcircled{e}_s\}$ for each $s \in S$, and
- $(TF \oplus PF)_{\to \mathbf{b}} = \{\mathbf{true}\}$.

and $\Gamma_{(S,TF,PF)}$ contains the following conditional equations:
1. $(\forall X)(X \textcircled{e} X = \mathbf{true}) \Rightarrow (\sigma(X) \textcircled{e} \sigma(X) = \mathbf{true})$ for any *total* operation symbols σ and X a string of variables matching the arity of σ. [2]
2. $(\forall X, Y)(X \textcircled{e} Y = \mathbf{true}) \Rightarrow (X \textcircled{e} X = \mathbf{true})$.
3. $(\forall X, Y)(X \textcircled{e} Y = \mathbf{true}) \Rightarrow (X = Y)$.
4. $(\forall X)(\sigma(X) \textcircled{e} \sigma(X) = \mathbf{true}) \Rightarrow (X \textcircled{e} X = \mathbf{true})$ for any *total* or *partial* operation symbols.

The set $\Gamma_{(S,TF,PF)}$ corresponds to characteristic properties of the existence equality relation $\stackrel{e}{=}$ on terms. Thus the equations 1. and 4. above correspond to the fact that a term $\sigma(t_1, \ldots, t_n)$ is defined if and only if its immediate sub-terms $t_1, \ldots, t_n$ are defined when σ is total and only the implication from the left to the right holds when σ is partial, whilst the equations 2. and 3. correspond to the fact that $t \stackrel{e}{=} t'$ implies both that t and t' are defined and that they are equal in the ordinary sense.

[2] If $X = \{x_1, \ldots, x_n\}$ then $X \textcircled{e} X$ denotes the finite conjunction $(x_1 \textcircled{e} x_1) \wedge \cdots \wedge (x_n \textcircled{e} x_n)$.

3.2. *The translation of the sentences*

Given a **PA** signature (S, TF, PF), the sentence translation $\alpha_{(S,TF,PF)}$ maps each (S, TF, PF)-sentence to a $(S \cup \{\mathbf{b}\}, TF \oplus PF)$-sentence as follows:
- $\alpha_{(S,TF,PF)}(t \stackrel{e}{=} t') = (t \textcircled{e} t' = \mathbf{true})$ for any terms t and t' of the same sort.
- $\alpha_{(S,TF,PF)}$ commutes with all Boolean connectives, i.e. $\alpha_{(S,TF,PF)}(\rho_1 \wedge \rho_2) = \alpha_{(S,TF,PF)}(\rho_1) \wedge \alpha_{(S,TF,PF)}(\rho_2)$, etc.
- $\alpha_{(S,TF,PF)}((\forall X)\rho) = (\forall X)((X \textcircled{e} X) \Rightarrow \alpha_{(S,TF \cup X,PF)}(\rho))$.

When there is no confusion about the context signature (S, TF, PF) we may simply denote $\alpha_{(S,TF,PF)}$ by α.

3.3. *The translation of the models*

This can be defined as a functor $\beta_{(S,TF,PF)} : \mathbf{ALG}(S \cup \{\mathbf{b}\}, TF \oplus PF, \Gamma_{(S,TF,PF)}) \to \mathbf{PA}(S, TF, PF)$ (often denoted simply by β when there is no danger of confusion) as follows:
- For each $(S \cup \{\mathbf{b}\}, TF \oplus PF)$-algebra A satisfying $\Gamma_{(S,TF,PF)}$,
  - $\beta(A)_s = \{a \in A_s \mid A_{\textcircled{e}_s}(a, a) = A_{\mathbf{true}}\}$ for each sort $s \in S$,
  - $\beta(A)_\sigma(a) = A_\sigma(a)$ when $A_{\textcircled{e}_s}(A_\sigma(a), A_\sigma(a)) = A_{\mathbf{true}}$ for any operation symbol $\sigma \in TF_{w \to s} \cup PF_{w \to s}$.
  - $\beta(A)_\sigma(a)$ is undefined when $A_{\textcircled{e}_s}(A_\sigma(a), A_\sigma(a)) \neq A_{\mathbf{true}}$ for any operation symbol $\sigma \in TF_{w \to s} \cup PF_{w \to s}$.

  Note that if $\beta(A)_\sigma(a)$ is defined then $\beta(A)_\sigma(a) \in \beta(A)$.
- For each $(S \cup \{\mathbf{b}\}, TF \oplus PF)$-algebra homomorphism $h : A \to B$ we define the homomorphism of partial algebras $\beta(h) : \beta(A) \to \beta(B)$ defined by $\beta(h)(a) = h(a)$ for each $a \in \beta(A)$.

Proposition 3.1 *For each $(S \cup \{\mathbf{b}\}, TF \oplus PF)$-algebra homomorphism $h : A \to B$, $\beta(h)$ is indeed a homomorphism of partial algebras $\beta(A) \to \beta(B)$.*

Proof. We first need to show that for each $a \in \beta(A)$ we have that $\beta(h)(a) \in \beta(B)$. Since $\beta(h)(a) = h(a)$ we need to have that $B_{\textcircled{e}}(h(a), h(a)) = B_{\mathbf{true}}$. But $B_{\textcircled{e}}(h(a), h(a)) = h(A_{\textcircled{e}}(a, a))$ because h is an $(S \cup \{\mathbf{b}\}, TF \oplus PF)$-algebra homomorphism. By the assumption that $a \in \beta(A)$ we have that $A_{\textcircled{e}}(a, a) = A_{\mathbf{true}}$. By the homomorphism condition on h we have that $h(A_{\mathbf{true}}) = B_{\mathbf{true}}$, hence $B_{\textcircled{e}}(h(a), h(a)) = B_{\mathbf{true}}$ and thus $\beta(h)(a) \in \beta(B)$.

Now let us show that for any $\sigma \in PF_{w \to s}$ and any $a \in \beta(A)_w$ if $\beta(A)_\sigma(a)$ is defined then $\beta(B)_\sigma(h(a))$ is defined too. We have the following:

$B_{\textcircled{e}}(B_\sigma(h(a)), B_\sigma(h(a))) =$

$= h(A_{\textcircled{e}}(A_\sigma(a), A_\sigma(a)))$ (h is homomorphism)

$= h(A_{\mathbf{true}})$ ($\beta(A)_\sigma(a)$ is defined)

$= B_{\mathbf{true}}$ (h is homomorphism).

Hence, by definition, $\beta(B)_\sigma(h(a))$ is defined.

Finally, that $h(\beta(A)_\sigma(a)) = \beta(B)_\sigma(h(a))$, when $\beta(A)_\sigma(a)$ defined, holds as follows:

$h(\beta(A)_\sigma(a)) =$

$= h(A_\sigma(a))$ (definition of $\beta(A)_\sigma(a)$)

$= B_\sigma(h(a))$ (h is homomorphism)

$= \beta(B)_\sigma(h(a))$ (definition of $\beta(B)_\sigma(h(a))$).

□

The functoriality of β, i.e. that it preserves the composition of homomorphisms and the identities follows very easily from the fact that as functions there is no difference between h and $\beta(h)$ when neglecting that the domain and the codomain of $\beta(h)$ are subsets of the domain and the codomain, respectively, of h.

3.4. *The satisfaction condition*

The property that we are going to prove below is rather crucial for the good functioning of logical encodings in general, and of our encoding in particular. In institution theory this property is known as the 'satisfaction condition' [9,5] or as the 'representation condition' [12] for institution comorphisms.

Theorem 3.2 (Satisfaction condition) *For each* **PA** *signature* (S, TF, PF)*, for each* $(S \cup \{\mathbf{b}\}, TF \oplus PF)$*-algebra* A *satisfying* Γ*, and for each* (S, TF, PF)*-sentence* ρ

$$A \models \alpha(\rho) \text{ if and only if } \beta(A) \models \rho.$$

Proof. We do the proof by induction on the structure of ρ. For the base case let us consider an existence equation $t \stackrel{e}{=} t'$.

Let as us assume that $\beta(A) \models t \stackrel{e}{=} t'$. This implies that both $\beta(A)_t$ and $\beta(A)_{t'}$ are defined and $\beta(A)_t = \beta(A)_{t'}$. We use the following lemma:

Lemma 3.3 *For any* (S, TF, PF)*-term* t*, if* $\beta(A)_t$ *is defined then* $\beta(A)_t = A_t$.

By Lemma 3.3 it follows that $\beta(A)_t = A_t$ which implies that$A_t \in \beta(A)$ which means $A_{\textcircled{e}}(A_t, A_t) = A_{\mathbf{true}}$. From Lemma 3.3 we also have $\beta(A)_{t'} = A_{t'}$, hence $A_t = A_{t'}$. Thus $A_{\textcircled{e}}(A_t, A_{t'}) = A_{\textcircled{e}}(A_t, A_t) = A_{\mathbf{true}}$ which means $A \models (t \textcircled{e} t' = \mathbf{true})$.

For showing the implication in the other direction, we assume $A \models (t \textcircled{e} t' = \mathbf{true})$. This means $A_{\textcircled{e}}(A_t, A_{t'}) = A_{\mathbf{true}}$. Because $A \models (\forall X, Y)(X \textcircled{e} Y = \mathbf{true}) \Rightarrow (X \textcircled{e} X = \mathbf{true})$ it follows that $A_{\textcircled{e}}(A_t, A_t) = A_{\mathbf{true}}$. We use the following lemma:

Lemma 3.4 *For any* (S, TF, PF)*-term* t*, if* $A_{\textcircled{e}}(A_t, A_t) = A_{\mathbf{true}}$ *then* $\beta(A)_t$ *is defined.*

Hence $\beta(A)_t$ is defined and by Lemma 3.3 we have that $\beta(A)_t = A_t$. From $A_{\textcircled{e}}(A_t, A_{t'}) = A_{\mathbf{true}}$, because $A \models (\forall X, Y)(X \textcircled{e} Y = \mathbf{true}) \Rightarrow (X = Y)$ it follows that $A_t = A_{t'}$, and because $A_{\textcircled{e}}(A_t, A_t) = A_{\mathbf{true}}$ we obtain that $A_{\textcircled{e}}(A_{t'}, A_{t'}) = A_{\mathbf{true}}$ which by Lemmas 3.3 and 3.4 implies that $\beta(A)_{t'}$ is defined and $\beta(A)_{t'} = A_{t'}$. We thus have that both $\beta(A)_t$ and $\beta(A)_{t'}$ are defined and that $\beta(A)_t = A_t = A_{t'} = \beta(A)_{t'}$, which means precisely that $\beta(A) \models t \stackrel{e}{=} t'$.

There are two step cases in our structural induction. One of them is when ρ is the negation or the conjunction of sentences for which the conclusion holds. This case is immediate since α preserves the Boolean connectives. The other case, when $\rho = (\forall X)\rho'$ is more interesting. In this case we assume that the conclusion holds for ρ' and have to prove that

$A \models (\forall X)((X \textcircled{e} X = \mathbf{true}) \Rightarrow \alpha(\rho'))$ if and only if $\beta(A) \models (\forall X)\rho'$.

For the implication from the left to the right, let B' be any $(S, TF \cup X, PF)$-expansion of $\beta(A)$. This yields an expansion A' of A defined by $A'_x = B'_x$ for each $x \in X$. Note that $A' \models (X \textcircled{e} X = \mathbf{true})$, which implies $A' \models \alpha(\rho')$. Since $\beta(A') = B'$, by the induction hypothesis we obtain that $B' \models \rho'$.

For the implication from the right to the left, let A' be any expansion of A such that $A' \models (X \textcircled{e} X = \mathbf{true})$. Then $\beta(A')$ is an $(S, TF \cup X, PF)$-expansion of $\beta(A)$, and by the induction hypothesis it follows that $A' \models \alpha(\rho')$.

We still owe the proofs of Lemmas 3.3 and 3.4:

Proof of Lemma 3.3: By induction on the structure of t. Consider $t = \sigma(t_1, \ldots, t_n)$. If $\beta(A)_{\sigma(t_1,\ldots,t_n)}$ is defined then for each $i \in \{1, \ldots, n\}$ we have that $\beta(A)_{t_i}$ is defined and that $\beta(A)_{t_i} \in dom(\beta(A)_\sigma)$. By the induction hypothesis $\beta(A)_{t_i}$ defined implies $\beta(A)_{t_i} = A_{t_i}$. Moreover $A_{\textcircled{e}}(A_{t_i}, A_{t_i}) = A_{\mathbf{true}}$ since $A_{t_i} \in \beta(A)$. Because each $\beta(A)_{t_i} = A_{t_i} \in dom(\beta(A)_\sigma)$, by the definition of $dom(\beta(A)_\sigma)$ we have that $A_{\textcircled{e}}(A_\sigma(A_{t_i}), A_\sigma(A_{t_i})) = A_{\mathbf{true}}$. Hence $\beta(A)_\sigma(\beta(A)_{t_1}, \ldots, \beta(A)_{t_n}) = \beta(A)_\sigma(A_{t_1}, \ldots, A_{t_n}) = A_\sigma(A_{t_1}, \ldots, A_{t_n}) = A_{\sigma(t_1,\ldots,t_n)}$.

Proof of Lemma 3.4: By induction on the structure of t. Consider $t = \sigma(t_1, \ldots, t_n)$. If $A_{\textcircled{e}}(A_{\sigma(t_1,\ldots,t_n)}, A_{\sigma(t_1,\ldots,t_n)}) = A_{\mathbf{true}}$ then $A_{\textcircled{e}}(A_\sigma(A_{t_1}, \ldots, A_{t_n}), A_\sigma(A_{t_1}, \ldots, A_{t_n})) = A_{\mathbf{true}}$. Since $A \models (\forall X)(\sigma(X) \textcircled{e} \sigma(X) = \mathbf{true}) \Rightarrow (X \textcircled{e} X = \mathbf{true})$ it follows that $A_{\textcircled{e}}(A_{t_i}, A_{t_i}) = A_{\mathbf{true}}$ for each $i \in \{1, \ldots, n\}$. By the induction hypothesis this means that $\beta(A_{t_i})$ is defined for each $i \in \{1, \ldots, n\}$. By Lemma 3.3 it follows that $\beta(A_{t_i}) = A_{t_i}$ and by the definition of $dom(\beta(A)_\sigma)$ that $(\beta(A)_{t_1}, \ldots, \beta(A)_{t_n}) \in dom(\beta(A)_\sigma)$. Hence $\beta(A)_{\sigma(t_1,\ldots,t_n)}$ is defined. □

The following is an immediate important consequence of the satisfaction condition Thm. 3.2.

Corollary 3.5 *For any* **PA** *signature* (S, TF, PF) *and any sets of* (S, TF, PF)*-sentences* E *and* E' *we have that*

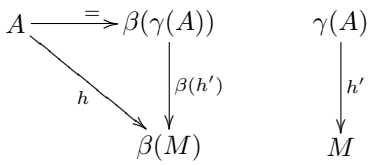
$E \models^{\mathbf{PA}}_{(S,TF,PF)} E'$ *implies* $\alpha(E) \cup \Gamma_{(S,TF,PF)} \models^{\mathbf{ALG}}_{(S \cup \{\mathbf{b}\}, TF \oplus PF)} \alpha(E')$.

4. Persistent liberality

In this section we show another important property for our encoding, namely that the model translations admit free constructions such that their universal homomorphisms are identities. We also develop a couple of important implications of this property.

Theorem 4.1 *For each* **PA** *signature* (S, TF, PF) *and for each partial* (S, TF, PF)*-algebra* A *there exists a* $((S \cup \{\mathbf{b}\}, TF \oplus PF), \Gamma_{(S,TF,PF)})$*-algebra* $\gamma(A)$ *such that*

$\beta(\gamma(A)) = A$ *and such that for each* $((S \cup \{\mathbf{b}\}, TF \oplus PF), \Gamma_{(S,TF,PF)})$*-algebra* M *and each partial algebra homomorphism* $h: A \to \beta(M)$ *there exists an unique* **ALG** *homomorphism* $h': \gamma(A) \to M$ *such that* $h = \beta(h')$.



Proof. Let $(S, TF + PF + A)$ denote the **ALG** signature with sorts S and such that
- $(TF+PF+A)_{w\to s} = TF_{w\to s} \cup PF_{w\to s}$ if w is non-empty, and
- $(TF + PF + A)_{\to s} = TF_{\to s} \cup PF_{\to s} \cup A_s$.

Let A^* denote the initial $(S, TF+PF+A)$-algebra satisfying the equations

$$A_\sigma(a) = \sigma(a)$$

for each operation symbol $\sigma \in TF_{w\to s} \cup PF_{w\to s}$ and for each $a \in dom(A_\sigma)$ (when σ is total operation symbol we consider $dom(A_\sigma) = A_w$). Note that A^* can be regarded as consisting of the elements of A plus all terms t formed with non-constant operation symbols from TF and PF and with the expressions $\sigma(a)$ for $a \notin dom(A_\sigma)$ in the role of constants. Then $\gamma(A)$ is defined as follows:
- $\gamma(A)_s = A^*_s$ for each $s \in S$,
- $\gamma(A)_{\mathbf{b}} = \{1\} \cup \{(s, a, a') \mid a, a' \in A^*_s, \ a \neq a' \text{ or } a \notin A_s\}$,
- $\gamma(A)_\sigma = A^*_\sigma$ for any operation symbol σ from TF or PF,
- $\gamma(A)_{\mathbf{true}} = 1$, and
- $\gamma(A)_{\circledcirc_s}(a, a') = \begin{cases} 1 & \text{when } a = a' \in A_s, \\ (s, a, a') & \text{otherwise.} \end{cases}$

It is easy to check that $\gamma(A) \models \Gamma_{(S,TF,PF)}$ and that $\beta(\gamma(A)) = A$.

For showing the universal property of $\gamma(A)$ we consider any (total) $(S \cup \{\mathbf{b}\}, TF \oplus PF)$-algebra M satisfying $\Gamma_{(S,TF,PF)}$ and any partial algebra homomorphism $h: A \to \beta(M)$. This yields a $(S, TF + PF + A)$-algebra M_h as follows:
- $(M_h)_s = M_s$ for each sort symbol $s \in S$ and $(M_h)_\sigma = M_\sigma$ for each operation symbol σ in TF or in PF, and
- $(M_h)_a = h(a)$ for each $a \in A$.

Let us show that $M_h \models A_\sigma(a) = \sigma(a)$ for each $a \in dom(A_\sigma)$. We have that $(M_h)_{A_\sigma(a)} = h(A_\sigma(a))$. Since $a \in dom(A_\sigma)$ and because $h: A \to \beta(M)$ is a homomorphism of partial algebras we have that $h(a) \in dom(\beta(M)_\sigma)$ and that $h(A_\sigma(a)) = \beta(M)_\sigma(h(a))$. By the definition of β, $h(a) \in dom(\beta(M)_\sigma)$ also implies that $\beta(M)_\sigma(h(a)) = M_\sigma(h(a))$. Since $(M_h)_{\sigma(a)} = (M_h)_\sigma((M_h)_a) = M_\sigma(h(a))$ we obtain that $(M_h)_{A_\sigma(a)} = (M_h)_{\sigma(a)}$.

Because A^* is the initial $(S, TF + PF + A)$-algebra satisfying the above mentioned equations, let $h^*: A^* \to M_h$ be the unique $(S, TF + PF + A)$-algebra homomorphism. We define the homomorphism $h': \gamma(A) \to M$ as follows:
- $h'(a) = h^*(a)$ for any $a \in A^*$,
- $h'(1) = M_{\mathbf{true}}$,
- $h'(s, a, a') = M_{\mathbf{true}}$ when $a, a' \in A_s$ and $h(a) = h(a')$, and
- $h'(s, a, a') = M_{\circledcirc}(h^*(a), h^*(a'))$ when $h^*(a) \neq h^*(a')$ or $a \notin A_s$ or $a' \notin A_s$.

In order to complete the proof of our theorem we have to prove three things about h': that h' is a homomorphism, that $\beta(h') = h$, and the uniqueness property of h'.

For the homomorphism property of h' we check three different cases as follows:
- For any operation symbol σ from TF or PF and for any list a of appropriate arguments for $\gamma(A)_\sigma$ we have the following sequence of equalities: $h'(\gamma(A)_\sigma(a)) = h'(A^*_\sigma(a)) = h^*(A^*_\sigma(a)) = (M_h)_\sigma(h^*(a)) = M_\sigma(h^*(a)) = M_\sigma(h'(a))$.
- $h'(\gamma(A)_{\mathbf{true}}) = h'(1) = M_{\mathbf{true}}$.
- For any two elements $a, a' \in \gamma(A)_s$ for $s \in S$, we have that $h'(\gamma(A)_{\circledcirc}(a, a')) =$

$$= \begin{cases} h'(1) & \text{when } a = a' \in A_s, \\ h'(s, a, a') & \text{otherwise.} \end{cases}$$

$$= \begin{cases} M_{\mathbf{true}} & \text{when } a, a' \in A_s, h(a) = h(a') \\ M_{\circledcirc}(h^*(a), h^*(a')) & \text{otherwise.} \end{cases}$$

But when $a, a' \in A_s$ and $h(a) = h(a')$ we have that $M_{\circledcirc}(h^*(a), h^*(a')) = M_{\circledcirc}(h(a), h(a')) = M_{\circledcirc}(h(a), h(a)) = M_{\mathbf{true}}$, the last equality holding by the virtue of the fact that $h(a) \in \beta(M)$. Hence $h'(\gamma(A)_{\circledcirc}(a, a')) = M_{\circledcirc}(h^*(a), h^*(a'))$ which means $h'(\gamma(A)_{\circledcirc}(a, a')) = M_{\circledcirc}(h'(a), h'(a'))$.

In order to prove that $\beta(h') = h$ we consider any element $a \in A$. Then we have the following sequence of equalities: $\beta(h')(a) = h'(a) = h^*(a) = h(a)$. Hence $\beta(h') = h$.

For showing the uniqueness of h' first let us note that since $\beta(h') = h$ then for each $a \in A$ we need to have that $h'(a) = h(a)$. Then by the homomorphism property of h' for the operation symbols in TF or PF, by the initiality property of A^* we obtain that $h'(a) = h^*(a)$ for each $a \in A^*$. Moreover $h'(1) = M_{\mathbf{true}}$ by the homomorphism property of h' for **true** and $h'(s, a, a') = h'(\gamma(A)_{\circledcirc}(a, a')) = M_{\circledcirc}(h'(a), h'(a')) = M_{\circledcirc}(h^*(a), h^*(a'))$ by the homomorphism property of h' for $\circledcirc_s$. □

Besides its technical consequences that we will develop below, the proof of Thm. 4.1 gives a very intuitive representation of partial algebras as total algebras. For example, if we consider the rational numbers $\mathbb{Q}$ with division $\_/\_$ as a partial operation, then its 'totalization' $\mathbb{Q}^*$ as given by Thm. 4.1 adds to the set of the rational numbers, as 'error' values, the normal forms of all terms (in the sense of evaluating parts of them as much as possible) formed by the rational numbers and operation symbols, such that $2/0+1/2$, etc.

The following important consequence of Thm 4.1, a well known result in the theory of partial algebras establishing the initial semantics for Horn partial algebra specifications, has been developed at the very general level of abstract

institutions in [12] (see also [5]). Here we develop it within the framework of our encoding.

Corollary 4.2 *Each set of quasi-existence equations admits an initial partial algebra.*

Proof. Let E be any set of quasi-existence equations for a **PA** signature (S, TF, PF). Note that the translation by α of any quasi-existence equation $(\forall X)H \Rightarrow C$ is semantically equivalent (i.e. satisfied by the same class of algebras) to the conditional equation $(\forall X)(X \textcircled{e} X) \wedge \alpha(H) \Rightarrow \alpha(C)$. By using the very classical result of existence of initial (total) algebras for any set of conditional equations, let M be the initial algebra of $\alpha(E) \cup \Gamma_{(S,TF,PF)}$. We show that $\beta(M)$ is the initial partial algebra satisfying E.

That $\beta(M) \models E$ follows from $M \models \alpha(E)$ by the satisfaction condition Thm. 3.2. For any partial algebra B satisfying E, because $\beta(\gamma(B)) = B$, by the satisfaction condition Thm. 3.2 we obtain that $\gamma(B) \models \alpha(E)$. Since $\gamma(B) \models \Gamma_{(S,TF,PF)}$ too, let $h: M \to \gamma(B)$ be the unique **ALG** homomorphism. Then $\beta(h): \beta(M) \to B$ is a partial algebra homomorphism.

For showing that $\beta(h)$ is unique let us consider $g: \beta(M) \to B$. Since $\beta(M) = \beta(\gamma(\beta(M)))$ and since $\beta(M) \models E$, by the satisfaction condition Thm. 3.2 we have that $\gamma(\beta(M)) \models \alpha(E)$, hence $\gamma(\beta(M)) \models \alpha(E) \cup \Gamma_{(S,TF,PF)}$; let $p_M: M \to \gamma(\beta(M))$ be the unique **ALG** homomorphism given by the initiality property of M. Let $q_M: \gamma(\beta(M)) \to M$ be the unique **ALG** homomorphism such that $\beta(q_M) = 1_{\beta(M)}$. By the initiality of M we have that $p_M; q_M = 1_M$, hence by applying β to this equality we obtain that $\beta(p_M) = 1_{\beta(M)}$. Since $p_M; \gamma(g): M \to \gamma(B)$, by the initiality of M we have that $h = p_M; \gamma(g)$. This implies $\beta(h) = \beta(p_M); \beta(\gamma(g)) = g$. □

The following extension Cor. 3.5 is another important consequence of Thm. 4.1.

Corollary 4.3 *For any* **PA** *signature* (S, TF, PF) *and any sets* E *and* E' *of* (S, TF, PF)*-sentences,*

$E \models^{\mathbf{PA}}_{(S,TF,PF)} E'$ *if and only if*

$\alpha(E) \cup \Gamma_{(S,TF,PF)} \models^{\mathbf{ALG}}_{(S \cup \{\mathbf{b}\}, TF \oplus PF)} \alpha(E')$.

Proof. The implication from the left to the right is given by Cor. 3.5. For showing the implication from the right to the left we consider any partial (S, TF, PF)-algebra A such that $A \models^{\mathbf{PA}}_{(S,TF,PF)} E$. By Thm. 4.1 we have that $\beta(\gamma(A)) = A$, hence by the satisfaction condition Thm. 3.2 we obtain that $\gamma(A) \models^{\mathbf{ALG}}_{(S \cup \{\mathbf{b}\}, TF \oplus PF)} \alpha(E)$. Since $\gamma(A) \models^{\mathbf{ALG}}_{(S \cup \{\mathbf{b}\}, TF \oplus PF)} \Gamma_{(S,TF,PF)}$ we have that $\gamma(A) \models^{\mathbf{ALG}}_{(S \cup \{\mathbf{b}\}, TF \oplus PF)} \alpha(E) \cup \Gamma_{(S,TF,PF)}$. By the hypothesis this implies $\gamma(A) \models^{\mathbf{ALG}}_{(S \cup \{\mathbf{b}\}, TF \oplus PF)} \alpha(E')$. By the satisfaction condition Thm. 3.2 again we obtain $A = \beta(\gamma(A)) \models^{\mathbf{PA}}_{(S,TF,PF)} E'$. □

5. Proof theoretic consequences

The model theoretic properties of our encoding result into an important proof theoretic consequence: the ordinary Birkhoff proof calculus for total algebras may serve as a sound and complete proof calculus for quasi-existence equations in partial algebra. Since the former calculus can be mechanized rather efficiently by term rewriting, our results provide a framework for the execution of partial algebra specifications with quasi-existence equations by ordinary term rewriting.

For any **ALG** signature (S, F) by $\vdash^{\mathrm{e}}_{(S,F)}$ let us denote the proof theoretic consequence relation generated by the ordinary Birkhoff proof calculus for conditional equations.

Corollary 5.1 *For any* **PA** *signature* (S, TF, PF)*, any set* E *of quasi-existence equations, and any* ρ *quasi-existence equation, both for* (S, TF, PF)*,*

$E \models^{\mathbf{PA}}_{(S,TF,PF)} \rho$ *if and only if*

$\alpha(E) \cup \Gamma_{(S,TF,PF)} \vdash^{\mathrm{e}}_{(S \cup \{\mathbf{b}\}, TF \oplus PF)} \alpha(\rho)$.

Proof. By Cor. 4.3 we have that $E \models^{\mathbf{PA}}_{(S,TF,PF)} \rho$ if and only if $\alpha(E) \cup \Gamma_{(S,TF,PF)} \models^{\mathbf{ALG}}_{(S \cup \{\mathbf{b}\}, TF \oplus PF)} \alpha(\rho)$. The remark made within the proof of Cor. 4.2, namely that each quasi-existence equation gets mapped by α to a sentence equivalent to a conditional equation, allows the application of the Birkhoff completeness result [2,8,4,5] giving that $\alpha(E) \cup \Gamma_{(S,TF,PF)} \models^{\mathbf{ALG}}_{(S \cup \{\mathbf{b}\}, TF \oplus PF)} \alpha(\rho)$ if and only if $\alpha(E) \cup \Gamma_{(S,TF,PF)} \vdash^{\mathrm{e}}_{(S \cup \{\mathbf{b}\}, TF \oplus PF)} \alpha(\rho)$. □

Example 5.2 Let us illustrate how Cor. 5.1 may be used to perfom quasi-existence equational calculus by means of ordinary equational calculus. For this we consider the soundness of the following **PA** congruence rule:

$$\{t \stackrel{\mathrm{e}}{=} t', \sigma(t) \stackrel{\mathrm{e}}{=} \sigma(t)\} \vdash \{\sigma(t) \stackrel{\mathrm{e}}{=} \sigma(t')\}$$

for t and t' terms and σ partial operation symbol (for simplicity σ is assumed to take only one argument). By Cor. 5.1 we have to show that

$$\Gamma \cup \{t \textcircled{e} t' = \mathbf{true}, \sigma(t) \textcircled{e} \sigma(t) = \mathbf{true}\} \vdash^{\mathrm{e}} \sigma(t) \textcircled{e} \sigma(t') = \mathbf{true}$$

By Substitutivity we have that $\Gamma \vdash^{\mathrm{e}} (t \textcircled{e} t' = \mathbf{true}) \Rightarrow (t = t')$, by Modus Ponens and Symmetry this implies $\Gamma \cup \{t \textcircled{e} t' = \mathbf{true}\} \vdash^{\mathrm{e}} (t = t') \vdash^{\mathrm{e}} (t' = t)$. From this, by term rewriting we obtain $\sigma(t) \textcircled{e} \sigma(t') = \sigma(t) \textcircled{e} \sigma(t) = \mathbf{true}$.

6. Conclusions

We have defined a semantic translation of partial algebra to total algebra by encoding the existence equality relation as an algebraic operation and by providing an axiomatization for the latter. We have proved a satisfaction condition and a persistent free construction property for this translation and developed a couple of consequences of these results, the most important being a sound and complete equational calculus with total operations for quasi-existence equations. Another significant consequence is a rather intuitive representation of partial algebras as total algebras.

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